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Auslander Regularity of p-adic Distribution Algebras

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Abstract

This thesis studies the Fréchet-Stein structure of the locally analytic distribution algebra associated to a p-adic Lie group defined over a finite extension $L/\mathbb{Q}_p$. As a main result it is shown that the defining Banach algebras are Auslander regular rings with a global dimension bounded above by the dimension of the group. As immediate consequences the dimension theory and parts of the duality theory for coadmissible modules are generalized to Lie groups over L. As an application we prove that coadmissible modules coming from smooth or, more general, $U(\mathfrak{g})$-finite representations, are zero-dimensional.

The proof of the main result relies on the study of certain L-analytic versions of uniform groups and their distribution algebras. For the latter the regularity of the family of Banach algebras is reduced to the case of a small index and follows then by filtration methods.

Contents

1 Introduction and notations

Mainly motivated by a possible p-adic extension of the local Langlands program P. Schneider and J. Teitelbaum recently developed a theory of continuous representations of p-adic Lie groups (cf. [ST1-6]). A central achievement of this theory is the construction of the category of admissible locally analytic representations (cf. [ST5]). Given a Lie group G over a finite extension $L/\mathbb{Q}_p$ these are locally analytic G-representations in topological K-vector spaces (K/L being a complete and discretely valued coefficient field) satisfying a suitable finiteness condition. This category contains on the one hand all interesting examples (finite dimensional algebraic representations, smooth admissible representations etc.) and yet, is a managable abelian category. More precisely, it is anti-equivalent to a well-behaved category $\mathcal{C}_G$ of certain ("coadmissible") topological modules over the locally analytic distribution algebra $D(G,K)$, an analytic analogue of the group algebra over K. It is a key result in the construction of $\mathcal{C}_G$ that, when G is compact, the algebra $D(G,K)$ is Fréchet-Stein. This means it is a projective limit of noetherian Banach algebras with flat transition maps and resembles therefore, although in general noncommutative, a ring of global holomorphic functions on a Stein manifold.

Furthermore, Schneider/Teitelbaum prove (cf. [ST5]) that when $L = \mathbb{Q}_p$ and G is compact, the Fréchet-Stein structure of $D(G,K)$ has an additional regularity property: the defining Banach algebras are Auslander regular rings with a global dimension bounded above by the dimension of the manifold G. Hence, in this case the (in general non-noetherian) ring $D(G,K)$ is "close" to an Auslander regular ring which has important consequences. On the one hand, it implies the existence of a well-behaved dimension theory on $\mathcal{C}_G$. On the other hand, it allows to deduce important properties of the locally analytic duality functor (cf. [ST6]) e.g. its involutivity. Establishing this regularity property of $D(G,K)$ over arbitrary base fields L was posed in [ST5] as an open problem.

The present work is dedicated to this matter. In particular, it gives a positive result in the expected sense (Thm. 5.10):

Main result. *Let G be a compact locally L-analytic group. Then $D(G,K)$ has the structure of a K-Fréchet-Stein algebra where the corresponding Banach algebras are Auslander regular rings whose global dimension is bounded above by the dimension of G.*

The exhibited structure of $D(G,K)$ comes as usual as a quotient structure from $D(G_0,K)$ via G_0, the scalar restriction from L to $\mathbb{Q}_p$ of G. Here, the algebra $D(G_0, K)$ receives its Fréchet-Stein structure from a suitable choice of open normal uniform subgroup $H \subseteq G$.

As an immediate consequence of the main result we generalize the dimension theory on $\mathcal{C}_G$, developed in [ST5] over $\mathbb{Q}_p$, to Lie groups G defined over finite extensions $L/\mathbb{Q}_p$. In particular, any coadmissible module in $\mathcal{C}_G$ has a well-defined codimension bounded above $\dim_L G$ and comes equipped with a filtration of finite length by coadmissible submodules. As an application we show that modules coming from smooth or, more generally, $U(\mathfrak{g})$-finite admissible G-representations as in [ST1] have maximal codimension equal to $\dim_L G$.

As a second consequence we briefly indicate how key results of the duality theory on $\mathcal{C}_G$, obtained in [ST6] over $\mathbb{Q}_p$, generalize to arbitrary base fields. Hence also for locally L-analytic groups G, the duality functor (defined on the bounded derived category of $D(G, K)$-modules with coadmissible cohomology) is an anti-involution. Due to the presence of the codimension the category $\mathcal{C}_G$ is filtered by abelian subquotient categories and the functor is computed as a particular Ext-group on each of them.

A brief description of the individual parts of this work is as follows:

In section 2 results on uniform groups are collected. A locally L-analytic extension of the notion of uniformness is proposed which can be expressed in terms of the Lie algebra. It is proved that a compact locally L-analytic group G contains enough subgroups H of this type.

In section 3 the vanishing ideal $I(G_0, K)$ i.e. the kernel of the canonical quotient map $D(G_0, K) \to D(G, K)$ is studied. This is motivated by the fact that the usual Fréchet-Stein structure of $D(G, K)$ consists of the quotient Banach algebras

$$D_r(G, K) = D_r(G_0, K)/I_r(G_0, K)$$

where $I_r(G_0, K)$ equals the closure of $I(G_0, K)$ inside a defining Banach algebra $D_r(G_0, K)$ of $D(G_0, K)$. An explicit finite set of generators for $I(G_0, K)$ is computed lying in a scalar extension of the Lie algebra of G_0. For subgroups H of the above type the elements of $D(H_0, K)$ admit power series expansions due to Mahler series of locally analytic functions on H_0. In this situation the generators turn out to be certain log-series generating also the completed ideal $I_r(H_0, K)$ of $D_r(H_0, K)$.

A Banach algebra $D_r(H_0, K)$ is endowed with its norm filtration and the graded ring is explicitly known (cf. [ST5]). We prove that the induced filtration on $I_r(H_0, K)$ is good with respect to the exhibited generators. Thus, the calculation of their principal symbols yields the explicit shape of the graded quotient ring

$$gr^{\cdot}_r\, D_r(H, K) = gr^{\cdot}_r\, D_r(H_0, K)/gr^{\cdot}_r\, I_r(H_0, K).$$

For a sufficiently small index r this ring is a polynomial ring in finitely many variables over a field which implies, for general reasons, that $D_r(H, K)$ is an Auslander regular ring of global dimension $\leq \dim_L G$. Unfortunately, it becomes also clear that $gr^{\cdot}_r\, D_r(H, K)$ in general has nonzero nilpotent elements and therefore infinite global dimension. Hence, the filtration technique yields not enough information.

Section 4: Motivated by this pathology we study finite free ring extensions of the type $D(N, K) \subseteq D(H, K)$ where $N \subseteq H$ is a normal open subgroup. We prove that for sufficiently many quotient norms on $D(H, K)$ the inclusion of the closure $D_{(r)}(N, K) \subseteq D_r(H, K)$ is a finite free ring extension between noetherian rings of the same global dimension. Furthermore, $D_r(H, K)$ is Auslander regular precisely if this holds for $D_{(r)}(N, K)$. We also show that in the special case of a member $H^{(m)}$ of the lower p-series of the uniform pro$-p$ group H, the restriction of such a quotient norm on $D_r(H, K)$ to $D_{(r)}(H^{(m)}, K)$ can be expressed in terms of an orthogonal basis of the Banach space $D_{(r)}(H^{(m)}, K)$. The latter basis comes as usual from a choice of ordered topological generators for the uniform group $H^{(m)}$.

In section 5 the main result for a general compact locally L-analytic group G is proved. This is an immediate consequence once it is established for a subgroup H of the type considered above. The latter is achieved via reducing the regularity of $D_r(H, K)$ to the situation of a sufficiently small index r with the help of the results of section 4. Then the filtration method of section 3 applies.

In section 6 and 7 the dimension theory and parts of the duality theory on $\mathcal{C}_G$ are generalized to arbitrary base fields, with results as pointed out above. Much of this follows formally from [ST5], Sect. 8 using that [loc.cit.], Thm. 8.9 on the existence of the regular Fréchet-Stein structure over $\mathbb{Q}_p$ admits our main result as a generalization.

Notations: Throughout the work p stands for a fixed prime number and $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ denotes a chain of complete intermediate fields where $L/\mathbb{Q}_p$ is finite of degree n and K is discretely valued. The absolute value $|.|$

on $\mathbb{C}_p$ is normalized as usual by $|p| = p^{-1}$. Let $\mathfrak{o} \subseteq L$ denote the valuation ring and $\mathfrak{m} \subseteq \mathfrak{o}$ the maximal ideal. Denote by π resp. e a prime element resp. the absolute ramification index of K. Hence $|\pi| = p^{-1/e}$. Let k be the residue field of K.

Usually G denotes a compact d-dimensional locally L-analytic group and G_0 stands for its scalar restriction to $\mathbb{Q}_p$. Usually H denotes an open subgroup of G whose underlying topological group is uniform. The Lie algebras of G resp. G_0 are denoted by $\mathfrak{g}_L$ resp. $\mathfrak{g}_{\mathbb{Q}_p}$. We also fix an exponential map $\exp : \mathfrak{g}_L - -- > G$ for G. As introduced in [ST2] let $C^{an}(G, K)$ denote the locally L-analytic K-valued functions on G and let $D(G, K)$ be the algebra of locally analytic K-valued distributions on G.

Put $\kappa := 1$ resp. $\kappa := 2$ if p is odd resp. even. Finally, for a set of indeterminates $X_{ij},\ 1 \leq i \leq n,\ 1 \leq j \leq d$, we will sometimes abbreviate $X_{21}, ..., X_{nd}$ for the subset $X_{ij},\ 2 \leq i \leq n,\ 1 \leq j \leq d$.

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2 Uniform groups

2.1 Uniform groups, lower p-series and p-valuations

Recall that the *lower p-series* $(P_i(G))_{i \geq 1}$ of an arbitrary pro-p group G is defined via

$$P_1(G) := G,\ P_{i+1}(G) := \overline{P_i(G)^p[P_i(G), G]},\ i \geq 1$$

(topological closure in G). It consists of (topologically) characteristic subgroups of G and one has $P_{i+1}(G) \subseteq P_i(G)$. If G is (topologically) finitely generated then each of the $P_i(G)$ is open and the series constitutes a fundamental system of open neighbourhoods for $1 \in G$ ([DDMS], Prop. 1.16).

Recall the definition of a powerful resp. uniform pro-p group: A pro-p group G is called *powerful* if $G/\overline{G^p}$ resp. $G/\overline{G^4}$ is abelian in case p is odd resp. even. If G is additionally finitely generated, then $P_{i+1}(G) = P_i(G)^p = G^{p^i}$ according to [loc.cit.], Thm. 3.6. A pro-p group G is called *uniform* if it is finitely generated, powerful and its lower p-series $(P_i(G))_{i \geq 1}$ satisfies

$$(G : P_2(G)) = (P_i(G) : P_{i+1}(G))$$

for all $i \geq 1$. (Since the lower p-series of a finite powerful p-group becomes eventually trivial, a uniform group is never finite.) A uniform group has a unique locally $\mathbb{Q}_p$-analytic structure ([loc.cit.], Thm. 8.18, Thm. 8.36 and Cor. 9.5) whose dimension coincides with the minimal number of generators. Thus, uniform groups may be considered special compact locally $\mathbb{Q}_p$-analytic groups.

Any minimal set of topological generators $g_1, ..., g_d$ of a uniform group G is an *ordered* set of topological generators in the sense that every element $g \in G$ can be written as

$$g = g_1^{x_1} \cdots g_d^{x_d}$$

where the numbers $x_i \in \mathbb{Z}_p$ are uniquely determined. Given such a system for G the subgroup $P_{i+1}(G)$ has the ordered system $g_1^{p^i}, ..., g_d^{p^i}$ of topological generators ([loc.cit.], Thm. 3.6 together with the discussion in [loc.cit.], 4.2).

Finally, if G is uniform then each $P_i(G)$ is a uniform pro-p group itself and is of the same dimension as G ([loc.cit.], Thm. 3.6 and Prop. 4.4).

On the other hand, there is the notion of a *p-valuation* ω on an arbitrary given group G. This is a real valued function

$$\omega : G \setminus \{1\} \longrightarrow (1/(p-1), \infty)$$

satisfying

1. $\omega(gh^{-1}) \geq \min\ (\omega(g), \omega(h))$,
2. $\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h)$,
3. $\omega(g^p) = \omega(g) + 1$

for all $g, h \in G$ ([Laz], III.2.1.2). As usual one puts $\omega(1) = \infty$. A p-valuation gives rise to a natural filtration of G by subgroups which can be used to define a topology on G ([loc.cit.], II.1.1.5). A p-valued group (G, ω), complete with respect to this topology, is called *p-saturated* if any $g \in G$ such that $\omega(g) > p/(p-1)$ is a p-th power ([loc.cit.], III.2.1.6).

Now specialize to the case where G is a compact locally $\mathbb{Q}_p$-analytic group endowed with a p-valuation ω. It follows from [Laz], III.3.1.3/9 and III.3.2.1 that the topology on G is defined by ω. Thus, G is a pro-p group by [loc.cit.], II.2.1.3 that has no torsion because of 3. Furthermore, by completeness of G, G admits an *ordered basis* ([loc.cit.], III.2.2.5/6). In our case, this is an ordered set of $d :=\dim_{\mathbb{Q}_p} G \geq 1$ elements $h_1, ..., h_d$ in G with the property: the map

$$\mathbb{Z}_p^d \longrightarrow G,\ (x_1, ..., x_d) \mapsto h_1^{x_1} \cdots h_d^{x_d}$$

is a bijective global chart for the manifold G satisfying

$$\omega(h_1^{x_1} \cdots h_d^{x_d}) = \min_{i=1,\ldots,d} (\omega(h_i) + v_p(x_i)) \tag{1}$$

([loc.cit.] III.3.1.7). Here, v_p is the normalized p-adic exponent on $\mathbb{Z}_p$. In particular, the ordered elements $h_1, ..., h_d$ are an ordered set of topological generators of G.

In [ST5], Sect. 4 the authors introduce the class of compact locally $\mathbb{Q}_p$-analytic groups G carrying a p-valuation ω that satisfies the following additional axiom

(HYP) (G, ω) is $p-$saturated and the ordered basis $h_1, ..., h_d$ of G satisfies $\omega(h_i) + \omega(h_j) > p/(p-1)$ for any $1 \leq i \neq j \leq d$.

If $d = 1$ the second condition is redundant.

This class and the class of uniform groups are closely related (see also [loc.cit.], remark after Lem. 4.3). Recall that $\kappa = 1$ resp. $\kappa = 2$ if p is odd resp. even.

Proposition 2.1 *Let G be a uniform pro-p group of dimension d. Then G has a p-valuation ω satisfying* (HYP). *It is given as follows: for $g \in P_i(G) \setminus P_{i+1}(G)$ put $\omega(g) := i$ resp. $\omega(g) := i + 1$ in case $p \neq 2$ resp. $p = 2$. In particular, ω is integrally valued. Any ordered system of topological generators $h_1, ..., h_d$ is an ordered basis and satisfies*

$$\omega(h_1) = ... = \omega(h_d) = \kappa.$$

Conversely, if $p \neq 2$ then any compact locally $\mathbb{Q}_p$-analytic group with a p-valuation satisfying (HYP) *is a uniform pro-p group.*

Proof: Let G be a uniform pro-p group. Define a function

$$\omega : G \setminus \{1\} \longrightarrow (1/(p-1), \infty)$$

as stated in the proposition. If $d > 1$ then clearly $\omega(h_i) + \omega(h_j) > p/(p-1)$ for $i \neq j$. Moreover, $\omega(g) > p/(p-1)$ for $g \in G$ implies $g \in P_2(G)$ and by [DDMS], Lem. 3.4 the group $P_2(G)$ consists (as a set) precisely of the p-th powers of G. So for the first statement it remains to see that ω really is a p-valuation. Let us check the above axioms 1.-3.: since each $P_i(G)$ is a subgroup of G containing $P_{i+1}(G)$ the axiom 1. is satisfied. According to [loc.cit.], Lem. 4.10 the map $x \mapsto x^p$ is a bijection $P_i(G)/P_{i+1}(G) \to P_{i+1}(G)/P_{i+2}(G)$ for all i. Thus, 3. is satisfied.

Now the lower p-series of any pro-p group satisfies

$$[P_i(G), P_j(G)] \leq P_{i+j}(G)$$

for all $i, j \geq 1$ ([loc.cit.], Prop. 1.16) which gives 2. in case $p \neq 2$. Assume $p = 2$. Then there is the stronger relation

$$[P_i(G), P_j(G)] \leq P_{i+j+1}(G)$$

for all $i, j \geq 1$ which will be proved in a separate lemma right after this proof. Thus, 2. holds also for $p = 2$. So ω really is a p-valuation. Clearly, it is integrally valued. Any ordered set of (topological) generators $h_1, ..., h_d$ is an ordered basis and must lie in $P_1(G) \setminus P_2(G)$. This follows directly from the discussion in [loc.cit.], 4.2. In particular, $\omega(h_i) = \kappa$ for all i.

Conversely, assume $p \neq 2$ and let a compact locally $\mathbb{Q}_p$-analytic group G be given together with a p-valuation ω satisfying (HYP). One has $\omega([g, h]) > p/(p-1)$ for all $g, h \in G$ which follows from (HYP) and equation (1) above. Since G is p-saturated this implies $[G, G] \subseteq G^p$ i.e. G is powerful. It is also torsionfree: A pro-p group can have at most p-torsion which is impossible by 3.. Hence, G is a finitely generated powerful pro-p group without torsion and so, according to [loc.cit.], Thm. 4.5, G must be uniform. $\square$

The following lemma was used in the preceding proof.

Lemma 2.2 *Let $p = 2$ and G be a powerful pro-2 group that is finitely generated. Then*

$$[P_i(G), P_j(G)] \leq P_{i+j+1}(G)$$

for all $i, j \geq 1$.

Proof: Fix an arbitrary i and use induction on j. Abbreviate $P_i := P_i(G)$ etc. According to [DDMS], Thm. 3.6 each P_i is powerfully embedded in G (in the sense of [loc.cit.], Def. 3.1) and so

$$[P_i, P_1] = [P_i, G] \leq P_i^4 = P_{i+2}.$$

This starts the induction. Now assume we have proved $[P_i, P_j] \leq P_{i+j+1}$ for all $j = 1, ..., k-1$. We want to show $[P_i, P_k] \leq P_{i+k+1}$. We have $P_j(G/N) \simeq P_jN/N$ for all j and for any open normal subgroup $N \leq G$ by [loc.cit.], Prop. 1.16. Since G is finitely generated P_{i+k+1} is open normal in G. Thus, putting $N := P_{i+k+1}$ and $G' := G/N$ it suffices to show that $[P_i(G'), P_k(G')] = 1$. Note that, according to [loc.cit.], Prop. 3.2, the finite group G' is a powerful p-group and so $P_j(G') = G'^{p^j} = \{x^{p^j}, x \in G'\}$ for any $j \in \mathbb{N}$ by [loc.cit.], Thm. 2.7. Abbreviate $P'_i := P_i(G')$ etc. and note that $P'_{i+k+1} = 1$.

Now $[P'_i, P'_{k-1}]$ is central in G'. Indeed,

$$[P'_i, P'_{k-1}] \leq P'_{i+k}$$

by induction hypothesis and $[P'_{i+k}, G'] \leq P'_{i+k+1} = 1$. Moreover, it is also of exponent p since $[P'_i, P'_{k-1}]^p \leq P'^p_{i+k} = P'_{i+k+1} = 1$. Now write as usual $x^y := y^{-1}xy$ for elements $x, y \in G'$. Then, for $x \in P'_i, y \in P'_{k-1}$ the general commutator identity

$$[x, y^p] = [x, y]\,[x, y]^y \cdots [x, y]^{y^{p-1}}$$

(e.g. [loc.cit.], statement 0.2) gives $[x, y^p] = [x, y]^p = 1$. Since $P'^p_{k-1} = \{y^p, y \in P'_{k-1}\}$ this implies $[P'_i, P'_k] = [P'_i, P'^p_{k-1}] = 1$. □

We finally explain briefly how the lower p-series p-valuation ω on a uniform group G relates to the K-Fréchet-Stein structure of $D(G, K)$ (see [ST5] for all details). Choose a minimal number of topological generators $h_1, ..., h_d$ of G where $d = \dim_{\mathbb{Q}_p} G \geq 1$. From the above proposition we obtain that this is an ordered basis for the p-valued group (G, ω) with $\omega(h_i) = \kappa$ for all i. The induced global chart $\mathbb{Z}_p^d \longrightarrow G$ leads to an identification $C^{an}(G, K) \simeq C^{an}(\mathbb{Z}_p^d, K)$ of locally convex K-vectorspaces. Elements of the right-hand side admit Mahler expansions in the usual way ([ST5], Sect. 4). This translates by duality into the following fact:

Write $b_i := h_i - 1 \in \mathbb{Z}_p[G] \subseteq D(G, K)$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} \cdots b_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$. Then every $\lambda \in D(G, K)$ has a unique convergent expansion

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ such that, for any $0 < r < 1$, the set $\{|d_\alpha| r^{\kappa|\alpha|}\}_\alpha$ is bounded. Conversely, any such expansion is convergent in $D(G, K)$. The value $\mathbf{b}^\alpha(f) \in K$ where $f \in C^{an}(G, K)$ has Mahler expansion

$$f(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \binom{\mathbf{x}}{\alpha}, \;\; c_\alpha \in K$$

is given by $\mathbf{b}^\alpha(f) = c_\alpha$.

Having this there is a family of norms $\|.\|_r,\ 0 < r < 1$ on $D(G, K)$ defined on an expansion $\lambda = \sum_\alpha d_\alpha \mathbf{b}^\alpha$ via

$$\|\lambda\|_r := \sup_\alpha |d_\alpha| r^{\tau\alpha} = \sup_\alpha |d_\alpha| r^{\kappa|\alpha|}$$

where $\tau\alpha := \sum_i \omega(h_i)\alpha_i = \kappa|\alpha|$. It gives the original strong topology on $D(G,K)$ which is a Fréchet topology by [ST2]. The subfamily for $p^{-1} < r < 1,\ r \in p^{\mathbb{Q}}$ consists of multiplicative norms and induces even the structure of a K-Fréchet-Stein algebra on $D(G,K)$ in the sense of the abstract definition given in [ST5], Sect. 3. The defining K-Banach algebras $D_r(G,K)$ are simply the completions of $D(G,K)$ with respect to the norms $||.||_r,\ p^{-1} < r < 1,\ r \in p^{\mathbb{Q}}$. Thus, $D_r(G,K)$ is given as K-Banach space by all convergent series

$$\sum_{\alpha\in\mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ uniquely defined and $|d_\alpha| r^{\kappa|\alpha|} \to 0$.

2.2 Uniform groups and standard groups

In this subsection we prove that any compact locally L-analytic group G contains an open normal subgroup H whose scalar restriction is uniform and satisfies an additional property. This is closely related to the notion of a standard group.

In the literature there are two definitions of a standard locally L-analytic group which differ (in case $L = \mathbb{Q}_p$) at the prime $p = 2$. A locally L-analytic group G of $\dim_L G = d$ is called *standard* of *level* $h,\ h \in \mathbb{N}$ if it admits a global chart onto $(\mathfrak{m}^h)^d \subseteq L^d$ such that the group operation is given by a single power series without constant term and with coefficients in $\mathfrak{o}^d$ ([B-L], III.7.3. Def. 1). (Remark: It follows by a simple comparison of coefficients that the operation $g \mapsto g^{-1}$ is then given by a single power series of the same type, e.g. [DDMS], Prop. 13.16 (ii).) A standard group of level 1 will simply be called standard.

In case $L = \mathbb{Q}_p$ a locally $\mathbb{Q}_p$-analytic group of $\dim_{\mathbb{Q}_p} G = d$ is called *standard** if it admits a global chart onto $p^\kappa\mathbb{Z}_p^d$ such that the group operation is given by a single power series without constant term and with coefficients in $\mathbb{Z}_p^d$ ([DDMS], Def. 8.22).

Proposition 2.3 *A standard* group G over $\mathbb{Q}_p$ is uniform.*

Proof: This is [DDMS], Thm. 8.31. We sketch the details: if ψ is the corresponding global chart $G \to p^\kappa\mathbb{Z}_p^d$ put for $i \geq 1$ (if $p > 2$) or $i \geq 2$ (if $p = 2$)

$$G(i) := \psi^{-1}(p^i\mathbb{Z}_p^d).$$

The $G(i)$ form an open neighbourhood base of $1 \in G$. It is then easy to see that each $G(i)$ is an open subgroup of G and that ψ induces a surjective group homomorphism $G(i) \to p^i\mathbb{Z}_p^d/p^{i+1}\mathbb{Z}_p^d$ with kernel $G(i+1)$. It follows from this that G is pro-p. Furthermore, one may deduce that the map $g \mapsto g^p$ induces an isomorphism of groups from $G(i)/G(i+1)$ onto $G(i+1)/G(i+2)$ which implies straightforward that $G(i+1) = \overline{G(i)^p}$ for all i. From this one deduces that G is powerful, topologically finitely generated and that one has $P_i(G) = G(i+\kappa-1)$ for all i. By the above, $(G(i) : G(i+1)) = p^d$ and hence, G is uniform. □

Corollary 2.4 *Suppose G is a locally $\mathbb{Q}_p$-analytic group G which is standard* with respect to the global chart $\psi : G \to p^\kappa\mathbb{Z}_p^d$. Denote by e_i the i-th unit vector in $\mathbb{Z}_p^d$. The elements*

$$h_i := \psi^{-1}(p^\kappa e_i)$$

constitute an ordered system $h_1, ..., h_d$ of topological generators of G.

Proof: By the above proposition G is pro-p and therefore $P_2(G)$ equals the Frattini subgroup of G ([DDMS], Prop. 1.13). In the notation of the above proof $P_2(G) = G(1+\kappa)$ and so $G/P_2(G)$ is isomorphic to $p^\kappa\mathbb{Z}_p^d/p^{\kappa+1}\mathbb{Z}_p^d$ via ψ. The quotient $G/P_2(G)$ is thus generated by the cosets of the $h_1, ..., h_d$ which implies that G is topologically generated by the $h_1, ..., h_d$ ([DDMS], Prop. 1.9). As explained previously, $h_1, ..., h_d$ is then an ordered system of topological generators. □

Lemma 2.5 *Fix $h \in \mathbb{N}$. Let H be a standard group and $\psi : H \longrightarrow \mathfrak{m}^d$ a corresponding global chart. Then $\psi^{-1}((\mathfrak{m}^h)^d)$ is an open normal subgroup of H and standard of level h.*

Proof: Let F be the power series with coefficients in $\mathfrak{o}^d$ and no constant term describing the group law on H with respect to the global chart ψ. Clearly, F evaluated on $(\mathfrak{m}^h)^d$ gives a group law on $(\mathfrak{m}^h)^d$ since $\mathfrak{m}^h$ is an ideal in $\mathfrak{o}$. Hence, $\psi^{-1}((\mathfrak{m}^h)^d)$ is an open subgroup H' of H. According to [B-L], III.7.4 Prop. 6 the commutator $[g, h]$ with $g \in H,\ h \in H'$ again lies in H'. Thus, H' is normal in H. Clearly, it is standard of level h. □

Lemma 2.6 *Let e' be the ramification index of $L/\mathbb{Q}_p$ and u a uniformizer for $\mathfrak{o}$. Let $k \in \mathbb{N},\ k \geq 2$ and let H be a locally L-analytic group of dimension*

d which is standard of level ke' with respect to a global chart ψ. Then H is standard of level 1 with respect to the global chart $\frac{1}{u^{ke'-1}} \cdot \psi$. Its scalar restriction H_0 is standard with respect to the global chart $H_0 \to (p^\kappa \mathfrak{o})^d \to p^\kappa \mathbb{Z}_p^{nd}$ where the first map equals $\frac{1}{p^{k-\kappa}} \cdot \psi$ and the second is induced by an arbitrary choice of $\mathbb{Z}_p$-basis for $\mathfrak{o}$.*

Proof: Denote the Lie algebra of H as usual by $\mathfrak{g}_L$. Then the global chart ψ induces an identification of $\mathfrak{g}_L$ with L^d as L-vectorspaces and we denote the induced L-basis of $\mathfrak{g}_L$ by $\mathfrak{y}_1, ..., \mathfrak{y}_d$. We thus regard ψ as a map $H \longrightarrow \Gamma$ where $\Gamma = \oplus_j \mathfrak{m}^{ke'} \mathfrak{y}_j \subseteq \mathfrak{g}_L$. By standardness we have for $g, h \in H$ and $\psi(g) = \sum_j \lambda_j \mathfrak{y}_j,\ \psi(h) = \sum_j \mu_j \mathfrak{y}_j$, with $\lambda_j, \mu_j \in \mathfrak{m}^{ke'}$ that

$$\psi(gh) = \sum_j F_j(\lambda_1, ..., \lambda_d, \mu_1, ..., \mu_d) \mathfrak{y}_j$$

where $F_j(X_1, ..., X_d, Y_1, ..., Y_d) \in \mathfrak{o}[[(X_s), (Y_s)]]$ without constant term. Hence, taking as L-basis of $\mathfrak{g}_L$

$$\mathfrak{x}_j := u^{ke'-1} \mathfrak{y}_j$$

we obtain $\Gamma = \oplus_j \mathfrak{m} \mathfrak{x}_j$ and for $g, h \in H$ and $\psi(g) = \sum_i \lambda_j \mathfrak{x}_j,\ \psi(h) = \sum_j \mu_j \mathfrak{x}_j$, with $\lambda_j, \mu_j \in \mathfrak{m}$ that

$$\psi(gh) = \sum_j F_j(\lambda_1 u^{ke'-1}, ..., \lambda_d u^{ke'-1}, \mu_1 u^{ke'-1}, ..., \mu_d u^{ke'-1}) \mathfrak{y}_j.$$

Since $F_j(X_1, ..., X_d, Y_1, ..., Y_d) \in \mathfrak{o}[[(X_s), (Y_s)]]$ has no constant term we obtain

$$\psi(gh) = \sum_j F'_j(\lambda_1, ..., \lambda_d, \mu_1, ..., \mu_d) \mathfrak{x}_j$$

where $F'_j(X_1, ..., X_d, Y_1, ..., Y_d) \in \mathfrak{o}[[(X_s), (Y_s)]]$ without constant term. By definition H is standard of level 1 with respect to the global chart

$$H \xrightarrow{\psi} \Gamma = \oplus_j \mathfrak{m} \mathfrak{x}_j \longrightarrow \mathfrak{m}^d.$$

It remains to see that H_0 is standard* with respect to the stated chart. For this note first that H is also standard of level $\kappa e'$. Indeed, take as an L-basis of $\mathfrak{g}_L$

$$\mathfrak{x}'_j := p^{k-\kappa} \mathfrak{y}_j.$$

Then

$$\Gamma = \oplus_j \mathfrak{m}^{ke'} \mathfrak{y}_j = \oplus_j \mathfrak{m}^{\kappa e'} \mathfrak{x}'_j$$

and for $g, h \in H$ and $\psi(g) = \sum_i \lambda_j \mathfrak{x}'_j,\ \psi(h) = \sum_j \mu_j \mathfrak{x}'_j$, with $\lambda_j, \mu_j \in \mathfrak{m}^{\kappa e'}$ we have

$$\psi(gh) = \sum_j F_j(\lambda_1 p^{k-\kappa}, ..., \lambda_d p^{k-\kappa}, \mu_1 p^{k-\kappa}, ..., \mu_d p^{k-\kappa})\mathfrak{y}_j.$$

Since $F_j(X_1, ..., X_d, Y_1, ..., Y_d) \in \mathfrak{o}[[(X_s),(Y_s)]]$ has no constant term and $k \geq 2$ (i.e. $p^{k-\kappa} \in \mathbb{Z}_p$) we get

$$\psi(gh) = \sum_j F''_j(\lambda_1, ..., \lambda_d, \mu_1, ..., \mu_d)\mathfrak{x}'_j \tag{2}$$

where $F''_j(X_1, ..., X_d, Y_1, ..., Y_d) \in \mathfrak{o}[[(X_s),(Y_s)]]$ without constant term. By definition H is standard of level $\kappa e'$ with respect to the global chart

$$H \xrightarrow{\psi} \Gamma = \oplus_j \mathfrak{m}^{\kappa e'} \mathfrak{x}'_j \longrightarrow (\mathfrak{m}^{\kappa e'})^d. \tag{3}$$

Choosing a $\mathbb{Z}_p$-basis $v_1, ..., v_n$ of $\mathfrak{o}$ yields $\mathfrak{m}^{\kappa e'} = \oplus_i p^\kappa \mathbb{Z}_p v_i$. From (3) together with the $\mathbb{Q}_p$-basis $\{v_i \mathfrak{x}'_j\}$ of $\mathfrak{g}_L$ we obtain the global chart

$$H_0 \xrightarrow{\psi} \Gamma = \oplus_j \oplus_i p^\kappa \mathbb{Z}_p v_i \mathfrak{x}'_j \longrightarrow (p^\kappa \mathbb{Z}_p)^{nd} \tag{4}$$

for the locally $\mathbb{Q}_p$-analytic group H_0. By (2) it follows for $g, h \in H_0$ and $\psi(g) = \sum_{ij} \lambda_{ij} v_i \mathfrak{x}'_j,\ \psi(h) = \sum_{ij} \mu_{ij} v_i \mathfrak{x}'_j$, with $\lambda_{ij}, \mu_{ij} \in p^\kappa \mathbb{Z}_p$ that

$$\begin{aligned}\psi(gh) = &\ \textstyle\sum_j F''_j((\sum_r \lambda_{rs} v_r)_s, (\sum_r \mu_{rs} v_r)_s)\mathfrak{x}'_j \\ = &\ \textstyle\sum_j \sum_i G_{ij}((\lambda_{rs}), (\mu_{rs})) v_i \mathfrak{x}'_j.\end{aligned}$$

Since $v_i \in \mathfrak{o}$ we have $v_i v_j = \sum_k c_{ijk} v_k$ with $c_{ijk} \in \mathbb{Z}_p$. Hence, the nd functions G_{ij} are given by power series with coefficients in $\mathbb{Z}_p$ and no constant term. By definition H_0 is therefore standard* with a global chart as claimed. □

It is known that a locally L-analytic group G has an open standard subgroup ([B-L], III.7.3 Thm. 4) and that a compact locally $\mathbb{Q}_p$-analytic group contains an open normal uniform subgroup ([DDMS], Cor. 8.34). The following result generalizes these two results.

Proposition 2.7 *Any compact locally L-analytic group G contains an open normal subgroup H which is standard. The corresponding global chart is given by*

$$H \xrightarrow{\exp^{-1}} \oplus_j \mathfrak{m}\mathfrak{x}_j \longrightarrow \mathfrak{m}^d$$

where exp *is the exponential map and $\mathfrak{x}_1, ..., \mathfrak{x}_d$ a suitable L-basis of $\mathfrak{g}_L$.*

Furthermore, the restricted group H_0 is standard and thus uniform.*

Proof: Choose a basis $\mathfrak{y}'_1, ..., \mathfrak{y}'_d$ of $\mathfrak{g}_L$, endow $\mathfrak{g}_L$ with the maximum norm and $\mathrm{End}_L(\mathfrak{g}_L)$ with the operator norm.

Choose $\lambda' \in L^\times$ and consider

$$\Lambda' := \oplus_j \lambda'^{-1} \mathfrak{m} \mathfrak{y}'_j.$$

If λ' has sufficiently big absolute value then the Hausdorff series converges on Λ' (viewed as a subset of L^d via the basis $\mathfrak{y}'_1, ..., \mathfrak{y}'_d$) turning it into a locally L-analytic group. We obtain an isomorphism of locally L-analytic groups $\exp : \Lambda' \to G'$ onto an open subgroup G' of G ([S], Cor. 16.13).

Enlarging λ' if necessary we may assume, according to [B-L], III.4.4 Cor. 3, that the identity in $\mathrm{End}_L(\mathfrak{g}_L)$

$$\mathrm{Ad}\,(\exp(\mathfrak{x})) = \exp'(\mathrm{ad}\ \mathfrak{x}) \tag{5}$$

holds for all $\mathfrak{x} \in \Lambda'$. Here, $\mathrm{Ad}(g)$ is as usual the tangent map to conjugation with $g \in G$, $\mathrm{ad}\ \mathfrak{x} = [\mathfrak{x}, .]$ and $\exp'$ is an exponential map for the locally L-analytic group $\mathrm{Aut}_L(\mathfrak{g}_L)$ defined by

$$\exp'(F) = \sum_{k\geq 0} F^k/k! \in \mathrm{Aut}_L(\mathfrak{g}_L)$$

for F sufficiently close to $0 \in \mathrm{End}_L(\mathfrak{g}_L)$.

Since ad: $\mathfrak{g}_L \to \mathrm{End}_L(\mathfrak{g}_L)$ is continuous enlarging λ' once more we may assume that $|\mathrm{ad}\ \mathfrak{x}| < p^{-\frac{1}{p-1}}$ for all $\mathfrak{x} \in \Lambda'$. This implies, given $g = \exp(\mathfrak{x}) \in G'$ that

$$|\mathrm{Ad}(g)| = |\exp'(\mathrm{ad}\ \mathfrak{x})| = |\sum_{k\geq 0} (\mathrm{ad}\ \mathfrak{x})^k/k!| = 1.$$

Hence, given any $\mathfrak{x} \in \mathfrak{g}_L$ and $g \in G'$ we obtain $|\mathrm{Ad}(g)\mathfrak{x}| \leq |\mathfrak{x}|$. In particular, Λ' is $\mathrm{Ad}(g)$-stable for all $g \in G'$.

Next, let $\mathcal{R}'$ be a (finite) system of representatives for the cosets G/G'. Put

$$\Lambda := \cap_{g\in\mathcal{R}'} \mathrm{Ad}(g)\Lambda'.$$

It follows that Λ is $\mathrm{Ad}(g)$-stable for all $g \in G$. Indeed, take $g \in G$, $\mathfrak{x} \in \Lambda$. We have $\mathfrak{x} \in \mathrm{Ad}(h)\Lambda' = \mathrm{Ad}(h)\mathrm{Ad}(g')\Lambda'$ for any $h \in \mathcal{R}'$, $g' \in G'$. Thus, if $g'' \in \mathcal{R}'$ is given we may choose the elements h, g' such that $hg' = g^{-1}g''$. It follows that $\mathrm{Ad}(g)\mathfrak{x} \in \mathrm{Ad}(g'')\Lambda'$. This proves that Λ is $\mathrm{Ad}(g)$-stable for all $g \in G$.

Furthermore, $p^t\Lambda$ is a subgroup of Λ' when $t \in \mathbb{N}$ is big enough. To see this choose $t \in \mathbb{N}$ big enough such that

$$g \exp(\mathfrak{x}) g^{-1} = \exp(\mathrm{Ad}(g)\mathfrak{x}) \tag{6}$$

holds for all $\mathfrak{x} \in p^t\Lambda'$, $g \in \mathcal{R}'$. This is possible according to [B-L], III.4.4 Cor. 3 and the fact that $\mathcal{R}'$ is finite. Then $p^t\Lambda$ is stable under $*$, the group operation of Λ'. Indeed, if $\mathfrak{x}, \mathfrak{y} \in p^t\Lambda$ and $g \in \mathcal{R}'$ then writing $\mathfrak{x} = \mathrm{Ad}(g)\mathfrak{x}'$, $\mathfrak{y} = \mathrm{Ad}(g)\mathfrak{y}'$ with $\mathfrak{x}', \mathfrak{y}' \in p^t\Lambda'$ one calculates that

$$\begin{aligned}\exp(\mathfrak{x} * \mathfrak{y}) = \exp(\mathrm{Ad}(g)\mathfrak{x}')\exp(\mathrm{Ad}(g)\mathfrak{y}') &= g\exp(\mathfrak{x}')\exp(\mathfrak{y}')g^{-1} \\ &= \exp(\mathrm{Ad}(g)(\mathfrak{x}' * \mathfrak{y}')).\end{aligned}$$

Here, the last two equalities follow from (6) where we have $\mathfrak{x}' * \mathfrak{y}' \in p^t\Lambda'$ since $p^t\Lambda'$ is a subgroup of Λ' ([S], Cor. 15.8). The calculation implies that $\mathfrak{x} * \mathfrak{y} \in \mathrm{Ad}(g)p^t\Lambda'$ and since this holds for any $g \in \mathcal{R}'$ we have $\mathfrak{x} * \mathfrak{y} \in p^t\Lambda$. Since $p^t\Lambda = -p^t\Lambda$ and $0 \in p^t\Lambda$ we obtain that $p^t\Lambda$ really is a subgroup of Λ' and therefore $\exp(p^t\Lambda)$ is an open subgroup of G.

Each $\mathrm{Ad}(g)$ is an L-linear isomorphism of $\mathfrak{g}_L$ and so $p^t\Lambda$ is an $\mathfrak{o}$-lattice in the L-vectorspace $\mathfrak{g}_L$ i.e. a free $\mathfrak{o}$-module of rank d. Thus, we may choose elements $\mathfrak{y}_1, ..., \mathfrak{y}_d$ in $\mathfrak{g}_L$ such that

$$p^t\Lambda = \oplus_j \mathfrak{m}\mathfrak{y}_j$$

and obtain the global chart

$$\varphi : \exp(p^t\Lambda) \to p^t\Lambda = \oplus_j \mathfrak{m}\mathfrak{y}_j \to \mathfrak{m}^d$$

for the open subgroup $\exp(p^t\Lambda)$ of G. According to (the proof of) [B-L], III.7.3 Thm. 4 we may pass to $\lambda^{-1}\mathfrak{m}^d$ for suitable $\lambda \in L \setminus \mathfrak{o}$ and obtain that the open subgroup

$$M := \exp(\lambda^{-1}p^t\Lambda)$$

of G is standard with respect to the global chart $\lambda \cdot \varphi|_M$.

Let $\mathcal{R}$ denote a system of representatives for G/M.

We now pass to

$$\Gamma := \oplus_j \lambda^{-1}\mathfrak{m}^{ke'}\mathfrak{y}_j \subseteq \oplus_j \lambda^{-1}\mathfrak{m}\mathfrak{y}_j = \lambda^{-1}p^t\Lambda$$

for suitable $k \in \mathbb{N}_0$, $k \geq 2$ which will be chosen conveniently in the following. Here, e' denotes the ramification index of L. In the rest of the proof we show that the open subgroup

$$H := \exp(\Gamma)$$

of G will satisfy our requirements.

Let us first check normality: by Lem. 2.5, H is an open normal subgroup of M. Enlarging k if necessary we may assume that

$$g\exp(\mathfrak{x})g^{-1} = \exp(\mathrm{Ad}(g)(\mathfrak{x})) \tag{7}$$

for all $\mathfrak{x} \in \Gamma$, $g \in \mathcal{R}$ (again by [B-L], III.4.4 Cor. 3) since $\mathcal{R}$ is finite. Now Λ is $\mathrm{Ad}(g)$-stable for all $g \in G$ and hence, so is $\Gamma = \lambda^{-1}u^{ke'-1}p^t\Lambda$ (u a uniformizer for $\mathfrak{o}$) since each $\mathrm{Ad}(g)$ is L-linear. Thus, identity (7) implies that $H = \exp(\Gamma)$ is stable under conjugation with elements from $\mathcal{R}$. Since H is also normal in M it follows that H is normal in G.

Now since M is standard with respect to the global chart $\lambda\cdot\varphi|_M$ it follows by Lem. 2.5 that H is standard of level ke' with respect to the global chart $\lambda \cdot \varphi|_H$ where we now recall that $\lambda \cdot \varphi|_H$ equals the map

$$\lambda \cdot \varphi|_H : H \stackrel{\lambda\cdot\exp^{-1}}{\longrightarrow} \lambda \cdot \Gamma = \oplus_j \mathfrak{m}^{ke'}\mathfrak{y}_j \longrightarrow (\mathfrak{m}^{ke'})^d.$$

This means precisely that for $g, h \in H$ and $\lambda \cdot \exp^{-1}(g) = \sum_i \lambda_j \mathfrak{y}_j$, $\lambda \cdot \exp^{-1}(h) = \sum_j \mu_j \mathfrak{y}_j$ with $\lambda_j, \mu_j \in \mathfrak{m}^{ke'}$ that

$$\lambda \cdot \exp^{-1}(gh) = \sum_j F_j(\lambda_1, ..., \lambda_d, \mu_1, ..., \mu_d)\mathfrak{y}_j$$

where $F_j(X_1, ..., X_d, Y_1, ..., Y_d) \in \mathfrak{o}[[(X_s), (Y_s)]]$ without constant term. By scaling H is standard of level ke' with respect to the chart

$$H \stackrel{\exp^{-1}}{\longrightarrow} \Gamma = \oplus_j \mathfrak{m}^{ke'}\mathfrak{y}'_j \longrightarrow (\mathfrak{m}^{ke'})^d$$

where $\mathfrak{y}'_j := \lambda^{-1}\mathfrak{y}_j$. Applying in this situation Lem. 2.6 we see that H is also standard of level 1 with a global chart given by

$$H \stackrel{\exp^{-1}}{\longrightarrow} \Gamma = \oplus_j \mathfrak{m}\mathfrak{x}_j \longrightarrow \mathfrak{m}^d$$

where $\mathfrak{x}_j := u^{ke'-1}\mathfrak{y}'_j$. Thus, the L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ is as desired.

Finally, Lem. 2.6 also implies that the restricted group H_0 is standard* and therefore uniform according to Prop. 2.3. □

Given a locally L-analytic group G a choice of L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$ gives rise to the map

$$\theta_L : (\sum_j x_j\mathfrak{x}_j) \mapsto \exp(x_1\mathfrak{x}_1) \cdots \exp(x_d\mathfrak{x}_d) \in G$$

defined on an open subset of $\mathfrak{g}_L = \oplus_j L\mathfrak{x}_j$ containing 0. It is locally L-analytic and étale at 0 and is called a *system of coordinates of the second kind* associated to the decomposition $\mathfrak{g}_L = \oplus_j L\mathfrak{x}_j$ ([B-L], III.4.3 Prop. 3).

Furthermore, the Lie algebras $\mathfrak{g}_L$ resp. $\mathfrak{g}_{\mathbb{Q}_p}$ of G resp. G_0 can be naturally identified as $\mathbb{Q}_p$-Lie algebras according to [B-VAR], 5.14.5.

Consider now the following condition on a subgroup H of G:

Condition (L)*: There is an L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$ and a $\mathbb{Z}_p$-basis $v_1, ..., v_n$ of $\mathfrak{o}$ with $v_1 = 1$ such that the system of coordinates of the second kind $\theta_{\mathbb{Q}_p}$ induced by the decomposition $\mathfrak{g}_{\mathbb{Q}_p} = \oplus_j \oplus_i \mathbb{Q}_p v_i \mathfrak{x}_j$ gives an isomorphism of locally $\mathbb{Q}_p$-analytic manifolds*

$$\theta_{\mathbb{Q}_p} : \oplus_j \oplus_i \mathbb{Z}_p v_i \mathfrak{x}_j \longrightarrow H_0.$$

The exponential satisfies $\exp(\lambda \cdot v_i \mathfrak{x}_j) = \exp(v_i \mathfrak{x}_j)^\lambda$ *for all $\lambda \in \mathbb{Z}$.*

We will refer to this condition several times in the following. Note that the last condition on the exponential is automatically satisfied when H is sufficiently small ([B-L], III.7.2 Prop. 3). Note also that if a subgroup $H \leq G$ is pro-p (e.g. uniform) and satisfies condition (L) with suitable bases v_i and $\mathfrak{x}_j$ then $\exp(\lambda \cdot v_i \mathfrak{x}_j) = \exp(v_i \mathfrak{x}_j)^\lambda$ for all $\lambda \in \mathbb{Z}$ extends to $\mathbb{Z}_p$-powers and so the elements

$$h_{ij} := \theta_{\mathbb{Q}_p}(v_i \mathfrak{x}_j) = \exp(v_i \mathfrak{x}_j)$$

are an ordered system of topological generators for H_0: every element $h \in H_0$ can be written as

$$h = \prod_j \prod_i h_{ij}^{x_{ij}}$$

where the numbers $x_{ij} \in \mathbb{Z}_p$ are uniquely determined.

As an obvious example the compact locally L-analytic group $(\mathfrak{o}, +)$ satisfies condition (L) and its scalar restriction to $\mathbb{Q}_p$ is uniform.

The above proposition gives

Corollary 2.8 *Any compact locally L-analytic group G has a fundamental system of open normal subgroups H such that H_0 is uniform and satisfies condition* (L)*.*

Proof: It suffices to show that the group H constructed in the proof of the last proposition is open normal in G with H_0 uniform satisfying (L). This is because $H \subseteq G'$ (in the notation of this proof) and by construction, the open subgroup G' of G can be chosen as small as desired. We use the notation of this proof.

According to it (and in connection with Lem. 2.6) H_0 is standard* with respect to the bijective global chart

$$\psi : H_0 \overset{\exp^{-1}}{\longrightarrow} \Gamma = \oplus_j \oplus_i p^\kappa \mathbb{Z}_p v_i \mathfrak{x}'_j \longrightarrow p^\kappa \mathbb{Z}_p^{nd}.$$

Here, $\mathfrak{x}'_1, ..., \mathfrak{x}'_d$ and $v_1, ..., v_n$ are bases of $\mathfrak{g}_L$ resp. $\mathfrak{o}$ and we may assume $v_1 = 1$ (compare line (4)). Denoting by e_{ij} the ij-th unit vector in $\mathbb{Z}_p^{nd}$ put

$$h_{ij} := \psi^{-1}(p^\kappa e_{ij}) = \exp(p^\kappa v_i \mathfrak{x}'_j)$$

for $i = 1, ..., n,\ j = 1, ..., d$. Cor. 2.4 implies that these nd elements are an ordered system $h_{11}, h_{21}, ..., h_{nd}$ of topological generators for H_0 and thus an ordered basis for the p-valued group H_0, according to Prop. 2.1. Hence, putting $\mathfrak{z}_j := p^\kappa \mathfrak{x}'_j$ it follows that the map

$$\theta_{\mathbb{Q}_p} : \oplus_j \oplus_i \mathbb{Z}_p v_i \mathfrak{z}_j \longrightarrow H_0,\ \sum_{ij} \lambda_{ij} v_i \mathfrak{z}_j \mapsto \prod_j \prod_i h_{ij}^{\lambda_{ij}}$$

where $\lambda_{ij} \in \mathbb{Z}_p$ is a locally $\mathbb{Q}_p$-analytic isomorphism. Since $h_{ij} = \exp(v_i \mathfrak{z}_j)$ it is by definition the system of coordinates of the second kind induced by the decomposition $\mathfrak{g}_{\mathbb{Q}_p} = \oplus_j \oplus_i \mathbb{Q}_p v_i \mathfrak{z}_j$.

It remains to check the last requirement of the condition (L). For fixed $\lambda \in \mathbb{Z}$ we have $\exp(\lambda \cdot \mathfrak{x}) = \exp(\mathfrak{x})^\lambda$ for all $\mathfrak{x} \in \Lambda'$ since the group law on Λ' is given by the Hausdorff series. Since $H = \exp(\Gamma)$ with $\Gamma \subseteq \Lambda'$ this identity holds a fortiori for all $\mathfrak{x} \in \Gamma$ and so in particular for $\mathfrak{x} = v_i \mathfrak{z}_j$.

All in all, H is an open normal uniform subgroup of G satisfying condition (L) with the bases $\mathfrak{z}_1, ..., \mathfrak{z}_d$ and $v_1, ..., v_n$. □

Corollary 2.9 *Suppose H is an open normal subgroup of G such that H_0 is uniform satisfying condition* (L). *Then any step in the lower p-series of H_0 endowed with the locally L-analytic structure as open subgroup of G is an open normal subgroup of G whose restriction is uniform satisfying condition* (L).

Proof: Let $H^{(m)}$ be the $(m+1)$-th step in the lower p-series of H_0 as topological group and give it the locally L-analytic structure as an open subgroup of G. It is thus open normal in G. Furthermore, its scalar restriction $(H^{(m)})_0$ equals $H_0^{p^m} = H_0^{(m)}$ as topological group and therefore as locally $\mathbb{Q}_p$-analytic group, according to [DDMS], Thm. 9.4. In particular, it is uniform. Let $\mathfrak{x}_1, ..., \mathfrak{x}_d$ resp. $v_1, ..., v_n$ be bases of $\mathfrak{g}_L$ resp. $\mathfrak{o}$ such that the induced coordinate system $\theta_{\mathbb{Q}_p}$ realizes condition (L) for H_0. In particular, the elements $h_{ij} := \theta_{\mathbb{Q}_p}(v_i \mathfrak{x}_j)$ are an ordered system $h_{11}, ..., h_{nd}$ of topological generators for H_0. Putting $\mathfrak{x}'_j := p^m \mathfrak{x}_j$ the $\mathbb{Q}_p$-basis $v_i \mathfrak{x}'_j$ of $\mathfrak{g}_{\mathbb{Q}_p}$ (which we identify with the Lie algebra of $H_0^{(m)}$ according to [B-L], III.3.8) is mapped by $\theta_{\mathbb{Q}_p}$ onto the ordered system of topological generators $h_{11}^{p^m}, ..., h_{nd}^{p^m}$ of $H_0^{(m)}$. Thus

$$\theta_{\mathbb{Q}_p} : \oplus_j \oplus_i \mathbb{Z}_p v_i \mathfrak{x}'_j \longrightarrow H_0^{(m)}$$

is a locally $\mathbb{Q}_p$-analytic isomorphism and by definition is the system of coordinates of the second kind induced by the basis $v_i\mathfrak{x}'_j$. Also,

$$\exp(\lambda \cdot v_i\mathfrak{x}'_j) = \exp(\lambda \cdot p^m v_i\mathfrak{x}_j) = \exp(v_i\mathfrak{x}_j)^{p^m\lambda} = \exp(v_i\mathfrak{x}'_j)^{\lambda}$$

for $\lambda \in \mathbb{Z}$. Hence, $H_0^{(m)}$ satisfies condition (L) with respect to the bases $\mathfrak{x}'_1, ..., \mathfrak{x}'_d$ and $v_1, ..., v_n$. $\square$

3 The vanishing ideal

Given a compact locally L-analytic group G with scalar restriction G_0 there is a canonical quotient map of K-Fréchet algebras $D(G_0, K) \to D(G, K)$. It arises by duality from the topological embedding of locally convex K-vector spaces

$$C^{an}(G, K) \subseteq C^{an}(G_0, K)$$

with closed image ([ST4], proof of Lem. 1.2). Its kernel is a two-sided and closed ideal

$$I(G_0, K) := \{\lambda \in D(G_0, K) : \lambda|_{C^{an}(G,K)} \equiv 0\}$$

which will be referred to as the *vanishing ideal* of $D(G_0, K)$.

3.1 Generators

It will be shown that the vanishing ideal $I := I(G_0, K)$ is finitely generated as a right ideal of $D(G_0, K)$. We explicitly determine a set of $nd - d$ generators F_{ij}, $2 \leq i \leq n$, $1 \leq j \leq d$. This will also be a set of generators for the right ideal I_r, the closure of I inside $D_r(G_0, K)$ where $D_r(G_0, K)$ denotes the completion of $D(G_0, K)$ along a defining norm for its Fréchet-Stein structure.

Whenever a locally convex K-vectorspace V is given we denote by V'_b its *strong dual* ([NFA], I.9.).

Lemma 3.1 *One has*

$$I'_b \simeq C^{an}(G_0, K)/C^{an}(G, K) \tag{8}$$

as locally convex K-vectorspaces where the right-hand side carries the quotient topology.

Proof: Note that
$$I \simeq (C^{an}(G_0,K)/C^{an}(G,K))'_b \tag{9}$$
as locally convex K-vector spaces where $I \subseteq D(G_0,K)$ carries the induced topology. Indeed, one has an exact sequence of locally convex K-vector spaces

$$0 \longrightarrow C^{an}(G,K) \longrightarrow C^{an}(G_0,K) \longrightarrow C^{an}(G_0,K)/C^{an}(G,K) \longrightarrow 0$$

which are of compact type ([ST2], Lem. 2.1 and Prop. 1.2.). Hence, by [ST2], Prop. 1.2 the induced sequence of strong duals is exact and the occuring maps are all strict. This gives (9). But being of compact type $C^{an}(G_0,K)/C^{an}(G,K)$ is reflexive ([loc.cit.], Thm. 1.1) and so the result follows. □

Now consider the Lie algebras $\mathfrak{g}_L$ resp. $\mathfrak{g}_{\mathbb{Q}_p}$ of G resp. G_0 and identify $\mathfrak{g}_L \simeq \mathfrak{g}_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$. Then exp may be viewed as an exponential map for $\mathfrak{g}_{\mathbb{Q}_p}$ as well. According to [ST2], Sect. 2 there is an action of $\mathfrak{g}_L$ on $C^{an}(G_0,K)$ via continuous endomorphisms given by

$$(\mathfrak{x}f)(g) := d/dt\, f(\exp(-t\mathfrak{x})g)\,|_{t=0}$$

for $\mathfrak{x} \in \mathfrak{g}_L,\ f \in C^{an}(G_0,K),\ g \in G$. Every $\mathfrak{x} \in \mathfrak{g}_L$ gives rise to the continuous linear form
$$f \mapsto (-\mathfrak{x}f)(1).$$
This induces an inclusion of $\mathbb{Q}_p$-vectorspaces $\mathfrak{g}_L \subseteq D(G_0,K)$ which extends to an L-linear inclusion

$$L \otimes_{\mathbb{Q}_p} \mathfrak{g}_L \subseteq D(G_0,K)$$

and allows to form the distributions

$$1 \otimes t\mathfrak{x} - t \otimes \mathfrak{x} \in D(G_0,K)$$

for $t \in L,\ \mathfrak{x} \in \mathfrak{g}_L$. Choose once and for all an L-basis $\mathfrak{x}_1,...,\mathfrak{x}_d$ of $\mathfrak{g}_L$ and a $\mathbb{Q}_p$-basis $v_1,...,v_n$ of L with $v_1 = 1$. Then $v_1\mathfrak{x}_1,...,v_n\mathfrak{x}_d$ is a $\mathbb{Q}_p$-basis of $\mathfrak{g}_L$.

Finally, denote by δ_g the Dirac distribution in $D(G_0,K)$ resp. $D(G,K)$ associated to $g \in G$.

Proposition 3.2 *The ideal I equals the closure of the K-vector space generated by the elements*

$$(1 \otimes t\mathfrak{x}_j - t \otimes \mathfrak{x}_j)\delta_g \in D(G_0,K)$$

where $t \in L,\ g \in G,\ j = 1,...,d$.

Proof: By [Ko], Lem. 1.3.2 the subspace $C^{an}(G, K)$ consists precisely of those $f \in C^{an}(G_0, K)$ for which the orbit map

$$\mathfrak{g}_L \longrightarrow C^{an}(G_0, K),\ \mathfrak{x} \mapsto \mathfrak{x}f$$

is not only $\mathbb{Q}_p$-linear but L-linear.

We put

$$M := \{f \in C^{an}(G_0, K) : t(\mathfrak{x}_j f) = (t\mathfrak{x}_j)f \text{ in } C^{an}(G_0, K) \text{ for all } \mathfrak{x}_j \text{ and } t \in L\}$$

and claim $M = C^{an}(G, K)$.

Clearly $C^{an}(G, K) \subseteq M$. Conversely take $f \in M$. Then for fixed $t \in L$

$$t((v_i\mathfrak{x}_j)f) = tv_i(\mathfrak{x}_j f) = (tv_i\mathfrak{x}_j)f$$

in $C^{an}(G_0, K)$ for all ij and so by $\mathbb{Q}_p$-linearity

$$t(\mathfrak{x}f) = (t\mathfrak{x})f$$

in $C^{an}(G_0, K)$ for all $\mathfrak{x} \in \mathfrak{g}_L$. Since $t \in L$ was arbitrary this means that the orbit map belonging to f is L-linear whence $f \in C^{an}(G, K)$.

Now consider the family of distributions

$$(1 \otimes t\mathfrak{x}_j - t \otimes \mathfrak{x}_j)\delta_g \in D(G_0, K)$$

with $t \in L,\ g \in G,\ j = 1, ..., d$. By the very definition of the inclusion $\mathfrak{g}_L \subseteq D(G_0, K)$ and by what we have just shown, the distributions $1 \otimes t\mathfrak{x}_j - t \otimes \mathfrak{x}_j$ vanish on the subspace $M = C^{an}(G, K)$ and so belong to I. Hence also $(1 \otimes t\mathfrak{x}_j - t \otimes \mathfrak{x}_j)\delta_g \in I$. But I is closed and so the closure of the K-vectorspace generated by this family, say W, is a subspace of I. In the rest of the proof we show that in fact $W = I$ whence the proposition follows.

Let $\phi \in I'_b$ be a continuous functional of the K-vectorspace I vanishing on the subspace W. We show $\phi \equiv 0$ whence $W = I$ by Hahn-Banach (e.g. [NFA], Cor. I.9.3).

By the above lemma we may write $\phi = \bar{f}$ for some $\bar{f} \in C^{an}(G_0, K)/C^{an}(G, K)$ and ϕ vanishing on W implies then

$$((1 \otimes t\dot{\mathfrak{x}}_j - t \otimes \dot{\mathfrak{x}}_j)\delta_g)(f) = 0.$$

Here, $\dot{}$ denotes "multiplication by -1" on $\mathfrak{g}_L$ and is put in here only to rule out the minus signs in the following. With the product formula for distributions (e.g. stated in [Ko], formula (1.2)) one may explicitly compute

$$\begin{aligned}
0 &= \delta_g(g' \longrightarrow (1 \otimes t\dot{\mathfrak{x}}_j - t \otimes \dot{\mathfrak{x}}_j)(g'' \longrightarrow f(g''g'))) \\
&= \delta_g(g' \longrightarrow (d/dt'\, f(\exp(-tt'\mathfrak{x}_j)g')\,|_{t'=0} - t\, d/dt'\, f(\exp(-t'\mathfrak{x}_j)g')\,|_{t'=0})) \\
&= \delta_g((t\mathfrak{x}_j)f - t(\mathfrak{x}_j f))
\end{aligned}$$

where in the last line $(t\mathfrak{x}_j)f - t(\mathfrak{x}_j f) \in C^{an}(G_0, K)$ (i.e. here, the action of $\mathfrak{g}_L$ on $C^{an}(G_0, K)$ is meant!). It follows

$$0 = ((t\mathfrak{x}_j)f - t(\mathfrak{x}_j f))(g)$$

for all $g \in G$ and so $(t\mathfrak{x}_j)f - t(\mathfrak{x}_j f) = 0$ as a function in $C^{an}(G_0, K)$. As this holds for all $j = 1, ..., d$ and $t \in L$ one has $f \in M = C^{an}(G, K)$. Hence, $\phi = \bar{f} = 0$. □

Now put for abbreviation

$$\partial_{ij} := v_i \mathfrak{x}_j \in \mathfrak{g}_L \subseteq D(G_0, K)$$

for the $\mathbb{Q}_p$-basis $v_i \mathfrak{x}_j$ of $\mathfrak{g}_L$ and so

$$1 \otimes v_i \mathfrak{x}_j - v_i \otimes \mathfrak{x}_j = \partial_{ij} - v_i \partial_{1j} \in D(G_0, K).$$

Proposition 3.3 *As a right ideal I is finitely generated by the family*

$$\partial_{ij} - v_i \partial_{1j} \in D(G_0, K)$$

where $i = 2, ..., n, \ j = 1, ..., d$.

Proof: From the proposition it follows by $\mathbb{Q}_p$-linearity that I is the closure of the K-vector space generated by the elements

$$(\partial_{ij} - v_i \partial_{1j})\delta_g$$

$i, j \geq 1$. Now δ_g is a unit and $v_1 = 1$. Thus, I is the closure of the right ideal generated by $\partial_{ij} - v_i \partial_{1j}$ where $i \geq 2, \ j \geq 1$. Since every finitely generated right ideal of $D(G_0, K)$ is already closed ([ST5], remark before Prop. 3.7, replace "left" by "right") the proposition follows. □

$D(G_0, K)$ is a K-Fréchet-Stein algebra. Let $||.||_r$ be one of the defining norms and let $D_r(G_0, K)$ be the completion. Let I_r be the closure of I in $D_r(G_0, K)$. This is again a two-sided ideal ([ST5], proof of Prop. 3.7).

Corollary 3.4 *As a right ideal I_r is finitely generated by the $nd - d$ elements*

$$\partial_{ij} - v_i \partial_{1j}$$

$i = 2, ..., n, \ j = 1, ..., d.$

Proof: The closure of I inside $D_r(G_0, K)$ equals $I\, D_r(G_0, K)$ ([ST5], proof of Prop. 3.7, replace "left" by "right"). □

Abbreviate for future reference

$$F_{ij} := \partial_{ij} - v_i \partial_{1j}$$

$i \geq 2, \ j \geq 1$ and call the family of these $nd - d$ elements $\mathcal{F}$.

3.2 Power series expansions

Recall that any compact locally L-analytic group G contains an open subgroup H whose scalar restriction H_0 is uniform satisfying (L) (Cor. 2.8). The results of the last section will be applied to such a group and a corresponding $\mathbb{Q}_p$-basis $v_i\mathfrak{x}_j$ of $\mathfrak{g}_{\mathbb{Q}_p}$ that realises condition (L) for H_0. The induced coordinate system $\theta_{\mathbb{Q}_p}$ identifies the manifolds H_0 and $\mathbb{Z}_p^{nd}$ and then elements of $D(H_0, K)$ admit unique canonical expansions coming via duality from Mahler expansions of locally analytic functions on $\mathbb{Z}_p^{nd}$. In this situation the expansions of the generators of $I(H_0, K)$ will be computed. These turn out to be certain logarithm series.

So let G be as usual a compact locally L-analytic group of dimension d. Let $\mathfrak{g}_L$ and $\mathfrak{g}_{\mathbb{Q}_p}$ be the Lie algebras of G and G_0. Consider an open subgroup H such that H_0 is uniform satisfying condition (L). According to [B-L], III.3.8 the Lie algebra $\mathfrak{g}_L$ can be naturally identified with the Lie algebra of H. Using this identification exp is an exponential for H as well according to [B-L], III.4.4 Prop. 8. Furthermore, $\mathfrak{g}_{\mathbb{Q}_p}$ identifies with the Lie algebra of H_0.

Now since H satisfies (L) we may choose an L-basis of $\mathfrak{g}_L$, a $\mathbb{Z}_p$-basis $v_1, ..., v_n$ of $\mathfrak{o}$ with $v_1 = 1$ such that the canonical coordinates of the second kind corresponding to the decomposition $\mathfrak{g}_{\mathbb{Q}_p} = \oplus_j \oplus_i \mathbb{Q}_p v_i \mathfrak{x}_j$ give an isomorphism

$$\theta_{\mathbb{Q}_p} : \oplus_j \oplus_i \mathbb{Z}_p v_i \mathfrak{x}_j \longrightarrow H_0$$

of locally $\mathbb{Q}_p$-analytic manifolds. Let $I := I(H_0, K)$ be the vanishing ideal of $D(H_0, K) \to D(H, K)$. Cor. 3.3 shows that as a right ideal of $D(H_0, K)$ it is generated by the $nd - d$ elements $F_{ij} := \partial_{ij} - v_i\partial_{1j}$ for $i \geq 2, j \geq 1$ where $\partial_{ij} = v_i\mathfrak{x}_j \in \mathfrak{g}_{\mathbb{Q}_p} \subseteq D(H_0, K)$.

Furthermore, put $h_{ij} := \theta_{\mathbb{Q}_p}(v_i\mathfrak{x}_j) = \exp(v_i\mathfrak{x}_j)$ for $i, j \geq 1$. Then the elements $h_{11}, h_{21}, ..., h_{nd}$ constitute an ordered set of topological generators for the uniform group H_0 which is, according to Prop. 2.1, at the same time an ordered basis for the p-valued group H_0. As explained in 2.1 the global chart

$$H_0 \xrightarrow{\theta_{\mathbb{Q}_p}^{-1}} \oplus_j \oplus_i \mathbb{Z}_p v_i \mathfrak{x}_j \longrightarrow \mathbb{Z}_p^{nd}$$

induces an identification $C^{an}(H_0, K) \simeq C^{an}(\mathbb{Z}_p^{nd}, K)$ of locally convex K-vectorspaces and therefore, via duality, expansions for the elements of $D(H_0, K)$ in the monomials $\mathbf{b}^\alpha = \prod_j \prod_i b_{ij}^{\alpha_{ij}}$ where $b_{ij} := h_{ij} - 1 \in \mathbb{Z}_p[H_0] \subseteq D(H_0, K)$, $\alpha \in \mathbb{N}_0^{nd}$.

Recall that $\log(1 + X) = \sum_{k \geq 1} (-1)^{k-1} X^k / k \in \mathbb{Q}[[X]]$.

Proposition 3.5 *Let $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$ and $t \in L$. Then the distribution*

$$1 \otimes t\mathfrak{x} - t \otimes \mathfrak{x} \in L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p} \subseteq D(H_0, K)$$

has the expansion

$$\sum_{i,j} (ST - t \cdot T)_{ij} \log(1 + b_{ij})$$

where $T = (t_{ij}) \in \mathrm{Mat}(\mathbb{Q}_p, n \times d)$, $t_{ij} \in \mathbb{Q}_p$ are the coefficients of $\mathfrak{x}$ with respect to the $\mathbb{Q}_p$-basis $v_i\mathfrak{x}_j$ of $\mathfrak{g}_{\mathbb{Q}_p}$ and $t \cdot T := (t\, t_{ij}) \in \mathrm{Mat}(L, n \times d)$. Moreover, $S \in \mathrm{Mat}(\mathbb{Q}_p, n \times n)$ is the representing matrix of "multiplication by t" on the $\mathbb{Q}_p$-vectorspace L.

Proof: Since the equality to prove is $\mathbb{Q}_p$-linear in $\mathfrak{x}$ we may assume that all $t_{ij} \in \mathbb{Z}_p$. Let $f \in C^{an}(H_0, K)$ be an analytic function. It has a Mahler expansion of the form

$$(f \circ \theta_{\mathbb{Q}_p})(\sum_{ij} x_{ij} v_i \mathfrak{x}_j) = \sum_{\alpha} c_\alpha \binom{\mathbf{x}}{\alpha}$$

for all $\mathbf{x} := (x_{11}, x_{21}, ..., x_{nd}) \in \mathbb{Z}_p^{nd}$ with unique coefficients $c_\alpha \in K$ and $|c_\alpha| s^{|\alpha|} \to 0$, $s > 1$ some real number. The $\mathbb{Q}_p$-linear inclusion $\mathfrak{g}_{\mathbb{Q}_p} \to D(H_0, K)$ maps a $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$ to the linear form

$$f \mapsto \mathfrak{x}(f) := \frac{d}{dt'} f(\exp(t'\mathfrak{x}))|_{t'=0}$$

(cf. 3.1). Thus, the value $(1 \otimes \mathfrak{x})(f) \in K$ is computed via

$$\begin{aligned}
(1 \otimes \mathfrak{x})(f) &= (\textstyle\sum_{ij} t_{ij} v_i \mathfrak{x}_j)(f) = \sum_{ij} (t_{ij} v_i \mathfrak{x}_j)(f) \\
&= \textstyle\sum_{ij} \frac{d}{dt'} f(\exp(t'(t_{ij} v_i \mathfrak{x}_j)))|_{t'=0} \\
&= \textstyle\sum_{ij} \frac{d}{dt'} f \circ \theta_{\mathbb{Q}_p}(t' t_{ij} v_i \mathfrak{x}_j)|_{t'=0} \\
&= \textstyle\sum_{ij} \sum_\alpha c_\alpha \frac{d}{dt'} \binom{t'\, t_{ij}}{\alpha_{ij}} \prod_{(i',j') \neq (i,j)} \binom{0}{\alpha_{i'j'}}|_{t'=0} \\
&= \textstyle\sum_{ij} \sum_\alpha c_\alpha \frac{d}{dt'} \binom{t'\, t_{ij}}{\alpha_{ij}} \delta_{\alpha,(0,\ldots,\alpha_{ij},0,\ldots)}|_{t'=0} \\
&= \textstyle\sum_{ij} \sum_{\alpha \neq 0} c_\alpha (-1)^{\alpha_{ij}-1}\, t_{ij}/\alpha_{ij}\; \delta_{\alpha,(0,\ldots,\alpha_{ij},0,\ldots)}.
\end{aligned}$$

The Kronecker delta $\delta_{\alpha,(0,\ldots,\alpha_{ij},0,\ldots)}$ is meant to be nonzero if and only if the index $\alpha \in \mathbb{N}^{nd}$ has only one non-zero entry α_{ij} at the coordinate ij. Hence

$$t \otimes \mathfrak{x} = t \sum_{ij} \sum_{k \geq 1} (-1)^{k-1} t_{ij}/k\; b_{ij}^k = \sum_{ij} (t \cdot T)_{ij} \log(1 + b_{ij}).$$

Now for the element $1 \otimes t\mathfrak{x} \in D(H_0, K)$: the ij-th coefficient of the element $t\mathfrak{x}$ with respect to the $\mathbb{Q}_p$-basis $\{v_i \mathfrak{x}_j\}$ equals the ij-th component $(ST)_{ij}$ of ST. Then the same computation as above gives the expansion

$$1 \otimes t\mathfrak{x} = \sum_{ij} (ST)_{ij} \log(1 + b_{ij}).$$

$\square$

The main result of this paragraph is

Proposition 3.6 *The generators F_{ij} of the ideal $I(H_0, K)$ have the expansions*

$$F_{ij} = \log(1 + b_{ij}) - v_i \log(1 + b_{1j}).$$

Proof: One has to compute the expansion of $1 \otimes v_i \mathfrak{x}_j - v_i \otimes \mathfrak{x}_j$. Letting $\mathfrak{x} = \mathfrak{x}_j$ and $t = v_i$ in the above proposition and looking at the corresponding matrices T, S one finds that the matrix T has $t_{1j} = 1$ and zeroes elsewhere and the first column of the matrix S has entry 1 in the i-th row and zeroes elsewhere (recall that $v_1 = 1$). Then the above proposition gives the claim. $\square$

3.3 Filtrations on the vanishing ideal

Let G be a compact locally L-analytic group. Any defining Banach algebra $D_r(G, K)$ for the Fréchet-Stein structure of $D(G, K)$ has a distinguished filtration via its norm. The main result of this section will be the explicit determination of the graded ring $gr_r^{\cdot} D_r(H, K)$ when H is an open subgroup of G such that H_0 is uniform and satisfies condition (L). It will turn out, provided that the radius $p^{-1} < r < 1$ is sufficiently close to p^{-1}, that this ring is essentially a polynomial ring in finitely many variables over the residue field k of K. In general, however, the ring $gr_r^{\cdot} D_r(H, K)$ contains nonzero nilpotent elements.

Throughout this subsection we fix an open subgroup H of G such that H_0 is uniform satisfying condition (L). Thus, we may fix a $\mathbb{Z}_p$-basis of $\mathfrak{o}$, say $v_1, ..., v_n$ with $v_1 = 1$ and an L-basis of $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$ such that the elements $h_{ij} := \exp(v_i \mathfrak{x}_j)$ are an ordered system $h_{11}, h_{21}, ..., h_{nd}$ of topological generators of H_0.

Put $b_{ij} := h_{ij} - 1 \in \mathbb{Z}_p[H]$ and $\mathbf{b}^\alpha := b_{11}^{\alpha_{11}} b_{21}^{\alpha_{21}} \cdots b_{nd}^{\alpha_{nd}}$ for $\alpha \in \mathbb{N}_0^{nd}$. Fix a norm $||.||_r$, $p^{-1} < r < 1$, $r \in p^{\mathbb{Q}}$ on $D(H_0, K)$ and denote by $D_r(H_0, K)$

the completion. As explained above the ring $D_r(H_0,K)$ equals a noncommutative power series ring in the "monomials" $\mathbf{b}^\alpha$. Since $||.||_r$ is multiplicative $D_r(H_0,K)$ is endowed with the filtration $(F_r^s D_r(H_0,K))_{s\in\mathbb{R}}$ defined by the additive subgroups

$$F_r^s D_r(H_0,K) := \{\lambda \in D_r(H_0,K),\ ||\lambda||_r \le p^{-s}\},$$

$$F_r^{s+} D_r(H_0,K) := \{\lambda \in D_r(H_0,K),\ ||\lambda||_r < p^{-s}\}.$$

This filtration is separated and exhaustive and $D_r(H_0,K)$ is complete with respect to it. It induces on K the usual filtration via the absolute value $|.|$. Since $r \in p^{\mathbb{Q}}$ and K is discretely valued the filtration is also quasi-integral (in the sense of [ST5], Sect. 1). Denote by

$$gr_r^{\cdot}\, D_r(H_0,K) := \oplus_{s\in\mathbb{R}} F_r^s D_r(H_0,K)/F_r^{s+} D_r(H_0,K)$$

the graded ring. Then Thm. 4.5 in [loc.cit.] states that there is an isomorphism of $gr^{\cdot} K$-algebras

$$gr_r^{\cdot}\, D_r(H_0,K) \simeq (gr^{\cdot} K)[X_{11},...,X_{nd}]$$

induced by mapping $\sigma(b_{ij}) \mapsto X_{ij}$ (σ denotes the principal symbol map). Moreover, let $\mathbb{F}_p[\epsilon]$ resp. $k[\epsilon_0,\epsilon_0^{-1}]$ denote the polynomial ring over $\mathbb{F}_p$ resp. the Laurent polynomials over k. Then it is easy to see that $\sigma(p) \mapsto \epsilon$ resp. $\sigma(\pi) \mapsto \epsilon_0$ induces isomorphisms

$$gr^{\cdot}\mathbb{Z}_p \simeq \mathbb{F}_p[\epsilon], \qquad gr^{\cdot} K \simeq k[\epsilon_0,\epsilon_0^{-1}]$$

as $\mathbb{F}_p$-algebras resp. k-algebras. The natural map $\mathbb{F}_p[\epsilon] \hookrightarrow gr^{\cdot} K$ arising from left exactness of the functor $gr^{\cdot}$ is then simply given by $\epsilon \mapsto a\epsilon_0^e$ where e denotes the absolute ramification index of K and $a \in k^\times$ is a suitable element.

The explicit descripition of $gr^{\cdot} K$ exhibits $gr_r^{\cdot}\, D_r(H_0,K)$ as a noetherian integral domain. Hence, by [LVO], Prop. II.2.2.1 $D_r(H_0,K)$ is a (left and right) Zariski ring.

For a nonzero $\lambda \in D_r(H_0,K)$ denote by $\deg(\lambda) \in \mathbb{R}$ the uniquely defined *degree* of λ in the filtration $F_r^{\cdot} D_r(H_0,K)$, i.e. $\deg(\lambda) = s$ if and only if $\lambda \in F_r^s D_r(H_0,K) \setminus F_r^{s+} D_r(H_0,K)$.

Recall that $I_r(H_0,K)$ is generated as a right ideal of $D_r(H_0,K)$ by the family

$$\mathcal{F} := \{F_{ij}\,,\ 2 \le i \le n,\ 1 \le j \le d\}$$

where $F_{ij} = \log(1+b_{ij}) - v_i \log(1+b_{1j})$. Endow $I_r(H_0,K) \subseteq D_r(H_0,K)$ and $D_r(H_0,K)/I_r(H_0,K)$ with the induced filtrations.

Finally, let us abbreviate $D_r := D_r(H_0,K)$, $I_r := I_r(H_0,K)$ for the rest of this section.

3.3.1 Principal symbols

In the following, the principal symbols $\sigma(F),\ F \in \mathcal{F}$ in the ring $\dot{gr_r}\, D_r$ will be computed for varying r.

First, recall some properties of the logarithm series

$$\log(1+X) := \sum_{k\geq 1} (-1)^{k-1} \frac{X^k}{k} \in \mathbb{Q}[[X]].$$

For every $0 < r < 1$ put $|\frac{X^k}{k}|_r := |\frac{1}{k}| r^k$ and consider the value

$$|\log(1+X)|_r := \sup_k |\frac{X^k}{k}|_r = \sup_k |\frac{1}{k}| r^k.$$

It follows from [R], VI.1.6 Ex. 2 that for $r < p^{-\frac{1}{p-1}}$ the linear term of $\log(1+X)$ is dominant, i.e. that

$$|X|_r > |\frac{X^k}{k}|_r$$

for all $k \geq 2$. According to [R], VI.1.4. the critical radii of the $\log(1+X)$-series, i.e. those radii such that there is no unique dominant monomial with respect to $|.|_r$ among the summands of $\log(1+X)$, are precisely of the form $p^{\frac{-1}{p^h - p^{h-1}}}$ for $h \in \mathbb{N}$. Finally, note that for r not critical the index of the dominant monomial in $\log(1+X)$ must be a p-power depending only on r. This is true since for any $h \in \mathbb{N}_0$ one has

$$|\frac{1}{p^h}| r^{p^h} > |\frac{1}{c}| r^c$$

for all $c \in \mathbb{N} \cap (p^h, p^{h+1})$. Recall that $\kappa = 1$ resp. $\kappa = 2$ if p is odd resp. even.

Lemma 3.7 *Let $p^{-1} < r < 1$ in $p^{\mathbb{Q}}$. Suppose r^κ is not a critical radius for $\log(1+X)$. In $\dot{gr_r}\, D_r = k[\epsilon_0, \epsilon_0^{-1}, X_{11}, ..., X_{nd}]$ one has*

$$\sigma(F_{ij}) = \epsilon^{-h} X_{ij}^{p^h} - \bar{v}_i \epsilon^{-h} X_{1j}^{p^h} \qquad (10)$$

with $h \in \mathbb{N}_0$ depending only on r^κ, $\bar{v}_i$ is the residue class of v_i in the residue field k of K and $\mathbb{F}_p[\epsilon] \subseteq k[\epsilon_0, \epsilon_0^{-1}]$ as explained above.

Proof: According to Prop. 3.6 we have

$$F_{ij} = \log(1 + b_{ij}) - v_i \log(1 + b_{1j}) \qquad (11)$$

in D_r. Now $||\log(1+b_{ij})||_r =: c$ is a constant for all ij and so

$$||F_{ij}||_r = ||\log(1+b_{ij})||_r \geq ||v_i \log(1+b_{1j})||_r. \qquad (12)$$

The principal symbol map is multiplicative on the product $v_i \log(1+b_{1j})$ since $gr^{\cdot}_r\, D_r$ is an integral domain. One gets

$$\sigma(F_{ij}) = \sigma(\log(1+b_{ij})) - \bar{v}_i \sigma(\log(1+b_{1j})) \qquad (13)$$

where $\bar{v}_i$ is the residue class of v_i in k. Indeed, if $|v_i| = 1$ then we have $\sigma(v_i) = \bar{v}_i$ and equality in (12). Moreover, since $\sigma(b_{ij})$ and $\sigma(b_{1j})$ are variables in the polynomial ring $gr^{\cdot}_r\, D_r$ the map σ is additive on the sum (11). Hence, (13) follows. If $|v_i| < 1$ then the inequality in (12) is strict and so $\sigma(F_{ij}) = \sigma(\log(1+b_{ij}))$ which is (13) since $\bar{v}_i = 0$.

It remains to compute $\sigma(\log(1+b_{ij}))$. Since r^κ is not a critical radius there is a dominant monomial of $\log(1+X)$ with respect to $|.|_{r^\kappa}$. By the above remark the corresponding index of this monomial must be a p-power, say $p^h,\ h \in \mathbb{N}_0$ and this h only depends on r^κ. Since

$$||\frac{b_{ij}^k}{k}||_r = |\frac{1}{k}|\, r^{\kappa k} = |\frac{X^k}{k}|_{r^\kappa}$$

for all ij and all k one gets

$$\sigma(\log(1+b_{ij})) = \sigma(\frac{b_{ij}^{p^h}}{p^h}) = \sigma(p^{-h}) X_{ij}^{p^h} = \epsilon^{-h} X_{ij}^{p^h}.$$

□

Corollary 3.8 *Let $p^{-1} < r < 1$ in $p^{\mathbb{Q}}$. Suppose r^κ is not a critical radius for* $\log(1+X)$. *The elements $F_{ij} \in \mathcal{F}$ are orthogonal in the normed K-vectorspace D_r, i.e. one has for arbitrary $c_{ij} \in K$ that*

$$||\sum_{ij} c_{ij} F_{ij}||_r = \max_{ij} ||c_{ij} F_{ij}||_r.$$

Proof: Given a sum $\sum_{ij} c_{ij} F_{ij}$ with $c_{ij} \in K$ we may (via leaving away possible summands) assume that $c_{ij} \neq 0$ for all ij and that $||c_{ij} F_{ij}||_r =: p^{-s},\ s \in \mathbb{R}$ is a constant for all ij. According to the above lemma we have

$$\sigma(F_{ij}) = \epsilon^{-h}\, X_{ij}^{p^h} - \bar{v}_i \epsilon^{-h}\, X_{1j}^{p^h}$$

with some $h \in \mathbb{N}_0$. Thus, the $\sigma(F_{ij})$ generate a free $gr^{\cdot} K$-module in the polynomial ring $gr^{\cdot}_r D_r = (gr^{\cdot} K)[X_{11}, ..., X_{nd}]$. We therefore get

$$0 \neq \sum_{ij} \sigma(c_{ij})\sigma(F_{ij}) = \sum_{ij} \sigma(c_{ij}F_{ij}) = \sum_{ij} c_{ij}F_{ij} \mod F_r^{s+} D_r$$

and so $|| \sum_{ij} c_{ij}F_{ij}||_r = p^{-s}$. $\square$

Lemma 3.9 *Let again $p^{-1} < r < 1$ in $p^{\mathbb{Q}}$. Assume that $r^\kappa < p^{-\frac{1}{p-1}}$. One has $h = 0$ in the notation of the preceding lemma, i.e.*

$$\sigma(F_{ij}) = X_{ij} - \bar{v}_i X_{1j}.$$

Proof: According to the remark before the preceding lemma our chosen r^κ is not critical and insures that the linear term of the series $\log(1 + X)$ is dominant with respect to $|.|_{r^\kappa}$. It follows $|\frac{X^k}{k}|_{r^\kappa} < |X|_{r^\kappa}$ for all $k \geq 2$ and so $h = 0$ in the notation of the preceding lemma. $\square$

Abbreviate $R := (gr^{\cdot} K)[X_{11}, ..., X_{nd}]$, $R' := (gr^{\cdot} K)[X_{11}, X_{12}, ..., X_{1d}]$ (so that $R = R'[X_{21}, ..., X_{nd}]$) and $Y_{ij} := X_{ij} - \bar{v}_i X_{1j}$ for all $i \geq 2,\ j \geq 1$.

Lemma 3.10 *The elements Y_{ij} are algebraically independent in R over the subring R'. The substitution homomorphism*

$$R'[X_{21}, ..., X_{nd}] \longrightarrow R'$$

induced by $X_{ij} \mapsto \bar{v}_i X_{1j}$ for all $i \geq 2, j \geq 1$ is surjective with kernel $(Y_{21}, ..., Y_{nd})$.

Proof: For any polynomial ring $A[Z_1, ..., Z_N]$ and elements $a_1, ..., a_N \in A$ the map $Z_i \mapsto Z_i - a_i$ is an automomorphism. Furthermore, the map $Z_i \mapsto a_i$ is surjective and has kernel $(Z_1 - a_1, ..., Z_N - a_N)$. Applying this to $R'[X_{21}, ..., X_{nd}]$ the result is clear. $\square$

Lemma 3.11 *Fix $h \in \mathbb{N}_0$. Consider the family $\mathcal{F}^\sigma$ in R consisting of the $nd - d$ elements*

$$\epsilon^{-h} X_{ij}^{p^h} - \bar{v}_i \epsilon^{-h} X_{1j}^{p^h},\ i \geq 2, j \geq 1$$

where as above $\mathbb{F}_p[\epsilon] \subseteq gr^{\cdot} K$. Take any t elements $\sigma_1, ..., \sigma_t$ in $\mathcal{F}^\sigma$. Denote by J the ideal in R generated by $\sigma_2, ..., \sigma_t$. Then the residue class $\overline{\sigma_1}$ is not a zero-divisor in R/J.

Proof: Since ϵ^{-h} is a unit in R and since we may choose elements $a_i \in k$ (the residue field of K) such that $a_i^{p^h} = \bar{v}_i$ the statement obviously follows from the following general fact: given any polynomial ring $A[Z_1, ..., Z_N]$ and nonzero integers $n_1, ..., n_k,\ k \leq N$ the elements $Z_1^{n_1}, Z_2^{n_2}, ..., Z_k^{n_k}$ (with any ordering) constitute a regular sequence on $A[Z_1, ..., Z_N]$ ([Ka], 3.1, Ex. 12 (c)). □

3.3.2 Good filtrations

Throughout the following a radius $r \in p^{\mathbb{Q}}$ with $p^{-1} < r < 1$ such that r^κ is not critical for $\log(1 + X)$, a norm $||.||_r$ on $D(H_0, K)$ and a completion $D_r := D_r(H_0, K)$ will be fixed. It will be shown that the induced filtration on $I_r := I_r(H_0, K)$ is good (in the sense of [LVO], Def. I.5.1) with respect to the generators $\mathcal{F}$. This implies that the graded module $gr^{\cdot}_r\, I_r$ is generated over $gr^{\cdot}_r\, D_r$ by the principal symbols $\sigma(F),\ F \in \mathcal{F}$. This allows us to determine the graded ring $gr^{\cdot}_r\, D_r / gr^{\cdot}_r\, I_r$ explicitly.

Recall our chosen bases $\mathfrak{x}_1, ..., \mathfrak{x}_d$ resp. $v_1, ..., v_n$ of $\mathfrak{g}_L$ over L resp. of $\mathfrak{o}$ over $\mathbb{Z}_p$ as introduced above. Consider the L-Lie algebra $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p}$. Put $\partial_{ij} := 1 \otimes v_i \mathfrak{x}_j \in L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p}$ as before. The family $\mathcal{F}$ of the $nd - d$ elements $F_{ij} := \partial_{ij} - v_i \partial_{1j},\ i \geq 2,\ j \geq 1$ generates $I(H_0, K)$ as a right ideal of $D(H_0, K)$. Define V to be the L-subspace generated by $\mathcal{F}$ inside $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p}$.

Lemma 3.12 *The subspace V is in fact an ideal in the Lie algebra $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p}$.*

Proof: It is easy to see that $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p}$ has the basis $\{\partial_{11}, \partial_{12}, ..., \partial_{1d}\} \cup \mathcal{F}$. Consider the homomorphism ϕ of L-Lie algebras $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p} \longrightarrow \mathfrak{g}_L$ induced by $v_i \otimes \mathfrak{x}_j \mapsto v_i \mathfrak{x}_j$ for all ij. Denote the kernel by W. Obviously $V \subseteq W$ since $\phi(F_{ij}) = \phi(v_i \otimes \mathfrak{x}_j) - \phi(1 \otimes v_i \mathfrak{x}_j) = 0$. Because of dimensions one must have $V = W$. □

For the following it is useful to make two definitions: Let $s \in \mathbb{R}$. An element $\lambda \in I_r$ can be *represented in* $F_r^s I_r$ (resp. $F_r^{s+} I_r$) if there are finitely many elements $F_k \in \mathcal{F},\ \lambda_k \in D_r$ such that $F_k \lambda_k \in F_r^s I_r$ (resp. $F_k \lambda_k \in F_r^{s+} I_r$) and

$$\lambda = \sum_k F_k \lambda_k.$$

Furthermore, let us say that a nonzero element $\lambda \in I_r$ has a *minimal representation* if λ can be represented in $F_r^s I_r$ where $s = \deg(\lambda)$.

Remarks:

1. In a minimal representation $\lambda = \sum_k F_k\lambda_k$ of an element $\lambda \in I_r$ all occuring summands $F_k\lambda_k$ have norm less than or equal to the norm of λ. This is the property which will be exploited in the following and explains the word "minimal".

2. For a nonzero element $\lambda \in I_r$ "having a minimal representation" is obviously equivalent to the property: for all $s \in \mathbb{R}$ with $\lambda \in F_r^s I_r$ it follows that λ can be represented in $F_r^s I_r$.

It will be shown below that every nonzero element of I_r has a minimal representation. The following lemma prepares this result.

Lemma 3.13 *Let $s \in \mathbb{R}$. Let a sum*

$$\sum_{k=1,\dots,t} F_k\lambda_k \quad \in F_r^{s+} I_r$$

with $t \geq 2$, $F_k \in \mathcal{F}$ (pairwise different), $\lambda_k \in D_r$ and $F_k\lambda_k \in F_r^s I_r$ be given. There are elements $\lambda'_1, \dots, \lambda'_{t-1} \in D_r$ with $F_k\lambda'_k \in F_r^s I_r$ and a term $T \in F_r^{s+} I_r$ such that

$$\sum_{k=1,\dots,t} F_k\lambda_k = T + \sum_{k=1,\dots,t-1} F_k\lambda'_k.$$

Moreover, T can be represented in $F_r^{s+} I_r$.

Proof: The proof is done in two steps: First, since the graded ring $\dot{gr_r}\, D_r$ is explicitly known, the statement for the sum $\sum F_k\lambda_k$ can be reduced to the statement for a "simpler" sum. Secondly, the preceding lemma will be applied to reduce the number of summands (modulo the term T) from t to $t-1$.

Abbreviate $R := \dot{gr_r}\, D_r$, $\|.\| := \|.\|_r$, $\sigma_k := \sigma(F_k) \in R$ for the principal symbol of $F_k \in \mathcal{F}$ and $c := \|F_k\|$. As explained in the proof of Lem. 3.7 the value c does not depend on k.

1. We may clearly assume that $\lambda_k \neq 0$ for all k. Then $F_k\lambda_k \neq 0$ for all k since D_r is an integral domain. Furthermore, we may assume that $\deg(F_k\lambda_k) = s$ for all k since other terms may finally be put into the term T. Then all $\sigma(F_k\lambda_k) = \sigma_k\sigma(\lambda_k)$ lie in the same homogeneous component of R and the assumption implies

$$\sum_{k=1,\dots,t} \sigma_k\sigma(\lambda_k) = 0$$

in R. Fix a k. Write J_k for the ideal generated by the elements $\sigma_1, ..., \widehat{\sigma_k}, ..., \sigma_t$ inside R ($\hat{}$ means "left out"). Hence, $\overline{\sigma_k}\,\overline{\sigma(\lambda_k)} = 0$ for the residue classes of σ_k and $\sigma(\lambda_k)$ in R/J_k. By Lem. 3.11 the residue class $\overline{\sigma_k}$ is not a zero divisor of R/J_k. Hence,

$$\sigma(\lambda_k) \in J_k,$$

say

$$\sigma(\lambda_k) = f_1^k \sigma_1 + ... + \widehat{f_k^k \sigma_k} + ... + f_t^k \sigma_t$$

with $f_i^k \in R$. Now the elements σ_i are homogeneous elements of the graded ring R, are of the same degree and the $f_i^k \sigma_i$ sum up to the homogeneous element $\sigma(\lambda_k)$. Thus, we may assume all $f_i^k,\ i = 1, ..., \hat{k}, ..., t$ to be homogeneous as well and of the same degree. This means that for all $i = 1, ..., \hat{k}, ..., t$ the degrees of $f_i^k \sigma_i$ and $\sigma(\lambda_k)$ coincide and there are elements $\lambda_i^k \in D_r$ such that

$$\sigma(\lambda_i^k) = f_i^k.$$

Letting now k vary one gets for all pairs (i, k) with $i \neq k$ that

$$\|F_i \lambda_i^k\| = \|\lambda_k\| = ||F_k||^{-1} p^{-s} = c^{-1} p^{-s} \tag{14}$$

and in particular $||\lambda_i^k|| = c^{-2} p^{-s}$.

Next, note that for fixed k

$$\sum_{i, i\neq k} \sigma_i \sigma(\lambda_i^k) = \sum_{i, i \neq k} \sigma_i f_i^k = \sigma(\lambda_k) \neq 0$$

and all terms in the sum of the left-hand side are of the same degree. So the principal symbol map is additive on this sum. Hence,

$$\sigma(\sum_{i, i\neq k} F_i \lambda_i^k) = \sigma(\lambda_k)$$

and so we may write

$$\lambda_k = \sum_{i, i \neq k} F_i \lambda_i^k + R_k$$

with some term $R_k \in D_r$ of norm strictly smaller than $\|\lambda_k\|$. Now put

$$A := \sum_{(k,i),\, k\neq i} F_k F_i \lambda_i^k, \qquad B := \sum_k F_k R_k$$

so that

$$\begin{aligned} \textstyle\sum_k F_k \lambda_k &= \textstyle\sum_{(k,i),\, k\neq i} F_k F_i \lambda_i^k &+& \textstyle\sum_k F_k R_k \\ &= A &+& B. \end{aligned}$$

Now $A = \sum_{k=1,\ldots,t} F_k \tilde{\lambda_k}$ with $\tilde{\lambda_k} := (\sum_{i,i\neq k} F_i \lambda_i^k)$ has the same shape as the sum $\sum_k F_k \lambda_k$, namely $A \in F_r^{s+} I_r$ but $F_k \tilde{\lambda_k} \in F_r^s I_r$. Moreover, since $\|F_k R_k\| < \|F_k \lambda_k\| = p^{-s}$ the term $B \in I_r$ is represented in $F_r^{s+} I_r$. Since the sum of two terms represented in $F_r^{s+} I_r$ is represented in $F_r^{s+} I_r$ it suffices to prove the statement of the lemma for the sum A.

2. The advantage of this is that in order to reduce the number of summands in A from t to $t-1$ (modulo a term of lower degree) we may now use the preceding lemma. First, write

$$A = \sum_{k,k<t} F_k (\sum_{i\neq k} F_i \lambda_i^k) + \sum_{i,i<t} F_t F_i \lambda_i^t. \tag{15}$$

According to the preceding lemma the L-subspace generated by the $F_{ij} \in \mathcal{F}$ in $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\mathbb{Q}_p}$ is an ideal and, in particular, stable under the Lie bracket. Thus

$$F_t F_i - F_i F_t = [F_t, F_i] = \sum_l c_l^i F_l'$$

with some pairwise different $F_l' \in \mathcal{F}$ and some coefficients $c_l^i \in L$. Since R is commutative the commutator of two elements has strictly smaller norm than their product and so

$$\|F_t F_i\| > \|[F_t, F_i]\| = \|\sum_l c_l^i F_l'\| = \sup_l \|c_l^i F_l'\|$$

where the last equality holds since the family $\mathcal{F}$ is orthogonal with respect to $||.||$ according to Cor. 3.8. It follows

$$\|c_l^i\| < c \tag{16}$$

for all i, l. Now, using $F_t F_i = F_i F_t + \sum_l c_l^i F_l'$ one gets

$$\begin{aligned} \textstyle\sum_{i,i<t} F_t F_i \lambda_i^t &= \textstyle\sum_{i,i<t} (F_i F_t + \sum_l c_l^i F_l') \lambda_i^t \\ &= \textstyle\sum_{i,i<t} F_i F_t \lambda_i^t + \sum_l F_l' (\sum_{i,i<t} c_l^i \lambda_i^t). \end{aligned}$$

So according to line (15) the term A can be rewritten as

$$\begin{aligned} A &= \textstyle\sum_{k<t} F_k (\sum_{i\neq k} F_i \lambda_i^k + F_t \lambda_k^t) &+ \textstyle\sum_l F_l' (\sum_{i,i<t} c_l^i \lambda_i^t) \\ &= \textstyle\sum_{k<t} F_k \lambda_k' &+ \textstyle\sum_l F_l' (\sum_{i,i<t} c_l^i \lambda_i^t) \end{aligned}$$

with

$$\lambda'_k := (\sum_{i\neq k} F_i\lambda_i^k + F_t\lambda_k^t).$$

Now $\|F_k\lambda'_k\| = c\ \|\lambda'_k\| \leq c\,c\,c^{-2}p^{-s} = p^{-s}$ according to (14) and so $F_k\lambda'_k \in F_r^s I_r$. Also,

$$\|F'_l \sum_{i,i<t} c_l^i\lambda_i^t)\| \leq \max_{i,i<t} \|c_l^i\|\ \|F'_l\lambda_i^t\| < c\ c^{-1}p^{-s} = p^{-s}$$

according to (14) and (16) and so the sum $\sum_l F'_l(\sum_{i,i<t} c_l^i\lambda_i^t)$ is represented in $F_r^{s+}I_r$. This proves the lemma. □

Proposition 3.14 *Every nonzero element of I_r has a minimal representation.*

Proof: Since K is discretely valued and $r \in p^{\mathbb{Q}}$ the filtration on I_r is quasi-integral with degrees in, say $1/q\cdot\mathbb{Z}$, $q\in\mathbb{N}$. Put $F^s := F_r^{s/q}I_r$ for $s\in\mathbb{Z}$.

Now let a nonzero $\lambda \in I_r$ be given. Since the family $\mathcal{F}$ generates I_r as a right ideal one has a representation

$$\lambda = \sum_{k=1,\ldots,t} F_k\lambda_k \tag{17}$$

with $F_k \in \mathcal{F}$ (pairwise different), $\lambda_k \in D_r$. There is $s\in\mathbb{Z}$ such that all terms $F_k\lambda_k \in F^s$, i.e. λ can be represented in F^s. In particular, $\lambda \in F^s$. We show that $\lambda \in F^{s+1}$ implies that λ can be represented in F^{s+1}. Then a (finite) induction on s yields that λ has a minimal representation. Assume $\lambda \in F^{s+1}$. If $t = 1$ then (17) is a representation in F^{s+1} and we are done. Hence, assume $t\geq 2$ in (17). Using the preceding lemma we may write

$$\lambda = T + \sum_{k=1,\ldots,t-1} F_k\lambda'_k$$

with some $\lambda'_k \in D_r$ such that $F_k\lambda'_k \in F^s$ and a term $T\in F^{s+1}$ that can be represented in F^{s+1}. By the ultrametric property $\sum_{k=1,\ldots,t-1} F_k\lambda'_k \in F^{s+1}$ and so the lemma applies to this latter sum as well. Since we have reduced the number of summands from t to $t-1$, repeating the argument finitely many times yields that λ can be represented in F^{s+1}. □

Pick $F\in\mathcal{F}$. Put $s_0 := \deg(F) \in \mathbb{R}$. This number is independent of the choice of F according to (the proof of) Lem. 3.7.

Corollary 3.15 *For any given $s \in \mathbb{R}$ one has*

$$F_r^s I_r = \sum_{F \in \mathcal{F}} F \; F_r^{s-s_0} D_r.$$

Proof: According to the proposition any element $\lambda \in F_r^s I_r$ can be represented in $F_r^s I_r$. Thus, $\lambda = \sum_k F_k \lambda_k$ with $F_k \lambda_k \in F_r^s I_r$ where $F_k \in \mathcal{F}$, $\lambda_k \in D_r$. The multiplicativity of the norm implies $\lambda_k \in F_r^{s-s_0} D_r$ and so $\lambda \in \sum_{F \in \mathcal{F}} F \; F_r^{s-s_0} D_r$. The reverse inclusion is trivial. □

The ring D_r is a complete filtered ring with (right) noetherian graded ring and therefore a (right) Zariski ring ([LVO], Thm. II.2.2.1). Hence, by [LVO], Thm. II.2.1.2 the induced filtration $F_r^{\cdot} I_r$ on I_r as a right D_r-module is good (in the sense of [LVO], Def. I.5.1). This implies by definition the existence of a family of elements in I_r, say $m_1, ..., m_l$ of degrees $s_1, ..., s_l \in \mathbb{R}$ in the filtration $F_r^{\cdot} I_r$ with the property

$$F_r^s I_r = m_1 F_r^{s-s_1} D_r + ... + m_l F_r^{s-s_l} D_r$$

for all $s \in \mathbb{R}$. These equations imply immediately that the graded module $gr_r^{\cdot} I_r$ is generated as a $gr_r^{\cdot} D_r$-module by the principal symbols of the m_i. The preceding corollary shows that $\mathcal{F}$ is such a family whence the main result of this paragraph follows:

Proposition 3.16 *The norm filtration on the D_r-module I_r is a good filtration with respect to the generators $\mathcal{F}$. Hence, the ideal $gr_r^{\cdot} I_r \subseteq gr_r^{\cdot} D_r$ is finitely generated by the $nd - d$ principal symbols $\sigma(F)$, $F \in \mathcal{F}$.*

3.3.3 Quotient rings

Since H_0 is uniform there is the usual family of norms $||.||_r$, $p^{-1} < r < 1$, $r \in p^{\mathbb{Q}}$ on $D(H_0, K)$. Consider the corresponding family of quotient norms $\|.\|_{\bar{r}}$ on $D(H, K)$ induced by the quotient map $D(H_0, K) \rightarrow D(H, K)$. According to [ST5], Prop. 3.7 and its proof this family induces on $D(H, K)$ the structure of a K-Fréchet-Stein algebra. The defining Banach algebras are the various completions $D_r(H, K)$ of $D(H, K)$. They are noetherian K-Banach algebras and canonically isomorphic to the quotients $D_r(H_0, K)/I_r(H_0, K)$. The norm $\|.\|_{\bar{r}}$ on $D_r(H, K)$ again induces a complete and separated filtration. We will compute the graded ring $gr_r^{\cdot} D_r(H, K)$ for varying r. Again, abbreviate $D_r := D_r(H_0, K)$, $I_r := I_r(H_0, K)$, $D_{\bar{r}} := D_r(H, K) = D_r/I_r$.

The $\|.\|_r$-filtration on D_r is quasi-integral since K is discretely valued and

$r \in p^{\mathbb{Q}}$. Hence by [B-CA], III.2.4 Prop. 2 the functor $gr^{\cdot}_r$ is exact on the short exact sequence

$$0 \longrightarrow I_r \longrightarrow D_r \longrightarrow D_{\bar{r}} \longrightarrow 0$$

of filtered D_r-modules when I_r carries the induced filtration and $D_{\bar{r}}$ has the $\|.\|_{\bar{r}}$-norm filtration. Hence,

$$gr^{\cdot}_r\, D_{\bar{r}} \simeq gr^{\cdot}_r\, D_r / gr^{\cdot}_r\, I_r$$

as $gr^{\cdot} K$-algebras. In particular, $D_{\bar{r}}$ is a (left and right) Zariski ring.

Proposition 3.17 *Suppose r^{κ} is not critical for* $\log(1+X)$. *Then there is a canonical isomorphism of $gr^{\cdot} K$-algebras*

$$gr^{\cdot}_r\, D_{\bar{r}} \xrightarrow{\sim} gr^{\cdot}_r\, D_r / gr^{\cdot}_r\, I_r \xrightarrow{\sim} (gr^{\cdot} K)[X_{11}, ..., X_{nd}] / (\{X_{ij}^{p^h} - \bar{v}_i X_{1j}^{p^h}\}_{i \geq 2, j \geq 1}),$$

the last map being induced by the isomorphism $gr^{\cdot}_r\, D_r \simeq (gr^{\cdot} K)[X_{11}, ..., X_{nd}]$. Here, $h \in \mathbb{N}_0$ depends only on r^{κ} and vanishes if $r^{\kappa} < p^{-\frac{1}{p-1}}$.

Proof: By Prop. 3.16 the ideal $gr^{\cdot}_r\, I_r$ is generated by the $nd - d$ symbols $\sigma(F_{ij}),\ i \geq 2,\ j \geq 1$. The isomorphism $gr^{\cdot}_r\, D_r(H_0, K) \simeq (gr^{\cdot} K)[X_{11}, ..., X_{nd}]$ identifies the latter with the elements $\epsilon^{-h}(X_{ij}^{p^h} - \bar{v}_i X_{1j}^{p^h})$ according to Lem. 3.7. Here, ϵ^{-h} is a unit and $h \in \mathbb{N}_0$ depends only on r^{κ}. By Lem. 3.9 one has $h = 0$ if $r^{\kappa} < p^{-\frac{1}{p-1}}$. □

Proposition 3.18 *Suppose that $r^{\kappa} < p^{-\frac{1}{p-1}}$. Then $gr^{\cdot}_r\, D_{\bar{r}}$ is isomorphic to a polynomial ring over $gr^{\cdot} K$ in d variables. The isomorphism of $gr^{\cdot} K$-algebras is given by*

$$gr^{\cdot}_r\, D_{\bar{r}} \simeq (gr^{\cdot} K)[X_{11}, ..., X_{nd}] / (\{X_{ij} - \bar{v}_i X_{1j}\}_{i \geq 2, j \geq 1}) \xrightarrow{\sim} (gr^{\cdot} K)[X_{11}, ..., X_{1d}]$$

where the last map is induced by the substitution homomorphism

$$X_{ij} \mapsto \bar{v}_i X_{1j}$$

for all $i \geq 2,\ j \geq 1$. The norm $\|.\|_{\bar{r}}$ is multiplicative and therefore $D_{\bar{r}}$ is an integral domain.

Proof: The first isomorphism follows from the preceding proposition since $r^{\kappa} < p^{-\frac{1}{p-1}}$ ensures that r^{κ} is not critical. Lem. 3.10 gives the second isomorphism. In particular, $gr^{\cdot}_r\, D_{\bar{r}}$ is an integral domain and so $\|.\|_{\bar{r}}$ has to be multiplicative. Then $D_{\bar{r}}$ must be an integral domain, too. □

Corollary 3.19 *The distribution algebra $D(H,K)$ is an integral domain.*

Proof: For any $p^{-1} < r < 1$ in $p^{\mathbb{Q}}$ one has the inclusion of rings $D(H,K) \subseteq D_r(H,K)$. If r is chosen with $r^\kappa < p^{-\frac{1}{p-1}}$ the right-hand side is an integral domain. $\square$

For the choice $H = (\mathfrak{o}, +)$ this was already shown in [ST4], Cor. 3.7.

Suppose that $r^\kappa < p^{-\frac{1}{p-1}}$ and put

$$b_j := b_{1j} \bmod I_r$$

in $D_r/I_r = D_{\bar{r}}$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} \cdots b_d^{\alpha_d} \in D_{\bar{r}}$ for an index $\alpha \in \mathbb{N}_0^d$. All these elements $\mathbf{b}^\alpha$ are nonzero. Indeed, $\prod_j b_{1j}^{\alpha_j} \in I_r$ implies $\prod_j X_{1j}^{\alpha_j} \in (X_{21} - \bar{v}_2 X_{11}, ..., X_{nd} - \bar{v}_n X_{1d})$ inside $gr^\cdot_r D_r$ by Prop. 3.16 which is a contradiction according to Lem. 3.10. Hence, we may consider the principal symbols $\{\sigma(\mathbf{b}^\alpha)\}_{\alpha \in \mathbb{N}_0^d}$ in $gr^\cdot_r D_{\bar{r}}$.

Proposition 3.20 *Suppose again that $r^\kappa < p^{-\frac{1}{p-1}}$. Every $\lambda \in D_{\bar{r}}$ has an expansion*

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with uniquely determined coefficients $d_\alpha \in K$ such that $\|d_\alpha \mathbf{b}^\alpha\|_{\bar{r}} \to 0$ for $|\alpha| \to \infty$. The multiplicative norm $\|.\|_{\bar{r}}$ is computed via

$$||\lambda||_{\bar{r}} = \sup_\alpha ||d_\alpha \mathbf{b}^\alpha||_{\bar{r}} = \sup_\alpha |d_\alpha| r^{\kappa\,|\alpha|}.$$

Proof: Let us prove the identity $||\mathbf{b}^\alpha||_{\bar{r}} = r^{\kappa|\alpha|}$ at the end. For the remaining statement it suffices to show that $gr^\cdot_r D_{\bar{r}}$ is freely generated as $gr^\cdot K$-module by the symbols $\{\sigma(\mathbf{b}^\alpha)\}_{\alpha \in \mathbb{N}_0^d}$. Indeed, the monomials $\mathbf{b}^\alpha$ will then generate a dense K-submodule in $D_{\bar{r}}$ such that $||\lambda||_{\bar{r}} = \max_\alpha ||d_a \mathbf{b}^\alpha||_{\bar{r}}$ for any finite sum $\lambda = \sum_\alpha d_\alpha \mathbf{b}^\alpha$ out of this submodule. It follows from this that every $\lambda \in D_{\bar{r}}$ has a convergent expansion $\sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$ as claimed with $||\lambda||_{\bar{r}} = \sup_\alpha ||d_\alpha \mathbf{b}^\alpha||_{\bar{r}}$. In particular, the coefficients $d_\alpha \in K$ are then uniquely determined.

So let us prove that $gr^\cdot_r D_{\bar{r}}$ is free as $gr^\cdot K$-module over the symbols $\{\sigma(\mathbf{b}^\alpha)\}_{\alpha \in \mathbb{N}_0^d}$. Since we already know that the norm $\|.\|_{\bar{r}}$ is multiplicative this amounts to prove that $gr^\cdot_r D_{\bar{r}}$ is a polynomial ring in the $\sigma(b_1), ..., \sigma(b_d)$ over $gr^\cdot K$. Write

$$\overline{X_{1j}} := X_{1j} \bmod gr^\cdot_r I_r$$

in $gr^\cdot_r D_r / gr^\cdot_r I_r = (gr^\cdot K)[X_{11}, ..., X_{nd}]/gr^\cdot_r I_r$. We know by Prop. 3.18 that these d elements generate $gr^\cdot_r D_r / gr^\cdot_r I_r$ as $gr^\cdot K$-algebra and are algebraically independent over $gr^{\cdot} K$. Hence, it suffices to check that the canonical isomorphism of $gr^{\cdot} K$-algebras

$$gr^{\cdot}_r D_r / gr^{\cdot}_r I_r \xrightarrow{\sim} gr^{\cdot}_r D_{\bar{r}} \tag{18}$$

maps $\overline{X_{1j}}$ to $\sigma(b_j)$ for all j. But this is clear: the isomorphism is induced by the graded morphism $gr\, f$ associated to the morphism $f : D_r \to D_{\bar{r}}$ of filtered D_r-modules. Thus, grf has the property that $gr\, f(\sigma(\lambda)) = \sigma(f(\lambda))$ for all $\lambda \in D_r$ as long as $f(\lambda)$ and $gr\, f(\sigma(\lambda))$ are nonzero. But, applying this to $\lambda := b_{1j}$ we already know that $b_j = f(\lambda)$ and $\overline{X_{1j}} = gr\, f(X_{1j}) = gr\, f(\sigma(\lambda))$ are nonzero. So the isomorphism (18) indeed maps $\overline{X_{1j}}$ to $\sigma(b_j)$.

Finally, the remaining identity $||\mathbf{b}^\alpha||_{\bar{r}} = r^{\kappa|\alpha|}$ reduces by multiplicativity of the norm $\|.\|_{\bar{r}}$ to the identity $\|b_j\|_{\bar{r}} = \|b_{1j}\|_r$. Suppose that $\deg(b_{1j}) = s$, i.e. $\sigma(b_{1j}) \in F^s D_r / F_r^{s+} D_r$. By what we have shown, we must have $\sigma(b_j) \in F^s D_{\bar{r}} / F_r^{s+} D_{\bar{r}}$ and thus $\deg(b_j) = s$ as well. □

Corollary 3.21 *Keeping the assumptions the isomorphism of Prop. 3.18*

$$gr^{\cdot}_r D_{\bar{r}} \xrightarrow{\sim} (gr^{\cdot} K)[X_{11}, ..., X_{1d}]$$

maps $\sigma(b_j) \mapsto X_{1j}$.

Proof: This was shown in the preceding proof. □

Remark: To finish this section we give a simple example to illustrate that $gr^{\cdot}_r D_{\bar{r}}$ in general has nonzero nilpotent elements and therefore infinite global dimension. Assume for simplicity that $L/\mathbb{Q}_p$ is a quadratic extension and that G has dimension $d = \dim_L G = 1$. Then $gr^{\cdot}_r D_r = (gr^{\cdot} K)[X_{11}, X_{21}]$ where $X_{11} = \sigma(b_{11}) \in gr^{\cdot}_r D_r$, $X_{21} = \sigma(b_{21}) \in gr^{\cdot}_r D_r$. Suppose that there is a $\mathbb{Z}_p$-basis of $\mathfrak{o}$ with $v_1 = 1$ and $|v_2| < 1$. For example, if $L/\mathbb{Q}_p$ is ramified such a basis always exists. Suppose that r is not critical for $\log(1 + X)$. With $X := X_{11}$, $Y := X_{21}$ we obtain by our above results that $gr^{\cdot}_r I_r = (Y^{p^h})$ and $gr^{\cdot}_r D_{\bar{r}} = (gr^{\cdot} K)[X, Y]/(Y^{p^h})$ where p^h is the index of the dominant monomial in the log-series. So if r is sufficiently close to 1 (recall that the critical radii are discrete in the unit interval) we certainly have $h > 0$ and then the residue class $\bar{Y} := Y \bmod (Y^{p^h})$ in $gr^{\cdot}_r D_{\bar{r}}$ is a nonzero nilpotent element.

A standard argument shows that gld $gr^{\cdot}_r D_{\bar{r}} = \infty$ in this case. Let $h > 0$ and put

$$R := (gr^{\cdot} K)[X, Y]/(Y^{p^h}) \quad \text{and} \quad M := (gr^{\cdot} K)[X, Y]/(Y).$$

M is an R-module via the obvious map $R \to M$. As such it has infinite projective dimension. Indeed, multiplying on R with a fixed power $\bar{Y}^s,\ s \in \mathbb{N}$ of the residue class $\bar{Y}$ induces an R-module endomorphism $\bar{Y}^s : R \to R$. With this notation there is a free resolution

$$... \stackrel{\bar{Y}^{p^h-1}}{\longrightarrow} R \stackrel{\bar{Y}}{\longrightarrow} R \stackrel{\bar{Y}^{p^h-1}}{\longrightarrow} R \stackrel{\bar{Y}}{\longrightarrow} R \longrightarrow M \longrightarrow 0$$

of M which is easily checked. Furthermore, no syzygy is projective according to the lemma below. Thus $\mathrm{pd}_R(M) = \infty$ as claimed and therefore $\mathrm{gld}(R) = \infty$.

Lemma 3.22 *Let $t \in \mathbb{N},\ t \geq 2$ and put $A := (gr^{\cdot} K)[X,Y],\ R := A/(Y^t)$. The R-module $\bar{Y}^s R$ is not projective for any $s \in \{1, ..., t-1\}$.*

Proof: Suppose $M := \bar{Y}^s R$ is projective. The sequence $R \stackrel{\bar{Y}^s}{\to} M \to 0$ splits and so $R = M \oplus N$ for some R-submodule N of R. Now $\bar{Y}^s N \subseteq N \cap M = 0$. Thus, if $1 = \bar{g_1} + \bar{g_2}$ with $\bar{g_1} \in M, \bar{g_2} \in N$ we have $\bar{Y}^s = \bar{Y}^s \bar{g_1}$. Choosing $g_1 \in Y^s A$ such that $g_1 \bmod Y^t = \bar{g_1}$ we therefore arrive at $Y^s = Y^{2s} f + Y^t h$ with $f, h \in A$. Since A is an integral domain this yields $1 = Y^s(f + Y^{t-s} h)$ in A and so $Y^s \in A^\times$ which is a contradiction to $s \geq 1$. $\square$

4 Direct sums and lower p-series

Given an arbitrary compact locally L-analytic group H such that H_0 is uniform the distribution algebra $D(H,K)$ is endowed with the usual family of quotient norms. In this section it is investigated how this family behaves under direct sum decompositions of $D(H,K)$ arising from a choice of open subgroup $N \leq H$. This is mainly motivated by the pathology of the norm filtration on the $D_r(H,K)$ (cf. last remark of Sect. 3).

4.1 Orthogonal bases

Let $(V, ||.||)$ be a normed K-vectorspace. Let I be a countable index set. A family of pairwise different nonzero elements $(v_i)_{i \in I}$ in V is called *orthogonal* if one has for any convergent series $v = \sum_I c_i v_i$ in V with $c_i \in K$ that

$$||v|| = \max_I |c_i| \, ||v_i||$$

([BGR], 2.7.2 Def. 6). The family is called an *orthogonal (Schauder) basis* if any element $v \in V$ can be written as such a convergent series. The coefficients $(c_i)_{i\in I}$ are then uniquely determined ([BGR], 2.7.2 Def. 1).

Now consider the uniform group H_0 and an ordered basis $h_1, ..., h_d$. Fix a norm $||.||_r$, $p^{-1} < r < 1$, $r \in p^{\mathbb{Q}}$ on $D(H_0, K)$ and consider the K-Banach space $D_r(H_0, K)$. According to 2.1 the elements $\{\mathbf{b}^\alpha\}_{\alpha\in\mathbb{N}_0^d}$ where $b_i := h_i - 1 \in \mathbb{Z}_p[H_0] \subseteq D_r(H_0, K)$ and $\mathbf{b}^\alpha = b_1^{\alpha_1} \cdots b_d^{\alpha_d}$ are an orthogonal K-basis.

Put a total ordering on $\mathbb{N}_0^d$ such that $\alpha \mapsto |\alpha|$ becomes order-preserving. For example we may use the ordering:

$\alpha \leq \beta$ if and only if the pair (α, β) satisfies one of the following conditions:

1. $|\alpha| < |\beta|$ or
2. $|\alpha| = |\beta|$ and $\alpha \leq \beta$ lexicographically.

The following lemma and its corollary will be used in 4.2 and 4.3.

Lemma 4.1 *Let $\mathcal{T} = \{T_i\}_{i\in I}$ be a countable family of pairwise different nonzero elements of $D_r(H_0, K)$. For any $T_i \in \mathcal{T}$ expand $T_i = \sum_{\alpha\in\mathbb{N}_0^d} t_{i,\alpha}\mathbf{b}^\alpha$. Choose an index $S_i \in \mathbb{N}_0^d$ such that $||T_i||_r = ||t_{i,S_i}\mathbf{b}^{S_i}||_r$ and such that S_i is maximal with respect to this condition. Suppose the following holds: for every finite set of pairwise different $T_1, ..., T_l$ in $\mathcal{T}$ there is $1 \leq k \leq l$ such that*

$$||t_{i,S_k}\mathbf{b}^{S_k}||_r < ||T_i||_r \tag{19}$$

for all $1 \leq i \leq l$ with $i \neq k$. Then $\mathcal{T}$ is an orthogonal family in $D_r(H_0, K)$. Suppose additionally that for any $\alpha \in \mathbb{N}_0^d$ there is $T_i \in \mathcal{T}$ such that $S_i = \alpha$. Then $\mathcal{T}$ is an orthogonal basis for $D_r(H_0, K)$.

Proof: [F], Lem. 1.4.1/2. □

Corollary 4.2 *Let $\mathcal{T} = \{T_i\}_{i\in I}$ be a countable family of pairwise different nonzero elements of $D_r(H_0, K)$. For any $T_i \in \mathcal{T}$ expand $T_i = \sum_{\alpha\in\mathbb{N}_0^d} t_{i,\alpha}\mathbf{b}^\alpha$ and choose an index $S_i \in \mathbb{N}_0^d$ with $||T_i||_r = ||t_{i,S_i}\mathbf{b}^{S_i}||_r$. Suppose the following holds: for each T_i the index S_i is uniquely determined and the elements of the family $\{S_i\}_{T_i\in\mathcal{T}}$ are pairwise different. Then the system $\mathcal{T}$ is orthogonal in $D_r(H_0, K)$.*

Suppose further that for any element $\alpha \in \mathbb{N}_0^d$ there is an element $T_i \in \mathcal{T}$ such that $S_i = \alpha$. Then $\mathcal{T}$ is an orthogonal basis for $D_r(H_0, K)$.

Proof: The first two assumptions imply that condition (19) in the above lemma is satisfied. Indeed, fix any $1 \leq k \leq l$ and choose some $1 \leq i \leq l$ with

$i \neq k$. Since S_i is uniquely determined we have

$$||t_{i,\alpha}\mathbf{b}^{\alpha}||_r < ||T_i||_r$$

for all $\alpha \in \mathbb{N}_0^d \setminus \{S_i\}$. Since $S_1, ..., S_l$ are pairwise different we may choose $\alpha = S_k$ and obtain

$$||t_{i,S_k}\mathbf{b}^{S_k}||_r < ||T_i||_r$$

which yields (19). Hence the above lemma gives the result. □

4.2 Direct sum decomposition

Any open subgroup H of an arbitrary compact locally L-analytic group G gives rise to a direct sum decomposition of $D(G, K)$ as a left $D(H, K)$-module (cf. [ST2], Sect. 2):

$$D(G, K) = \oplus_{g \in \mathcal{R}} D(H, K)g \tag{20}$$

where $\mathcal{R}$ is a system of representatives for $H \backslash G$. This decomposition is topological with respect to the Fréchet topologies on $D(G, K)$ resp. $D(H, K)$ and follows by duality from the product decomposition of locally convex K-vectorspaces

$$C^{an}(G, K) = \prod_{g \in \mathcal{R}} C^{an}(Hg, K)$$

([Fea], Kor. 2.2.4). Whenever $D(H, K)$ carries a multiplicative norm $||.||$ with $||h|| = 1$ for all $h \in H \subseteq D(H, K)$ one may define on $D(G, K)$ the maximum norm: put for $\sum_{g \in \mathcal{R}} \lambda_g \, g \in D(G, K), \ \lambda_g \in D(H, K)$

$$||\lambda|| := \max_g ||\lambda_g||.$$

It is independent of the choice of $\mathcal{R}$. This turns $D(G, K)$ into a normed left $D(H, K)$-module which is isometrically isomorphic to the normed direct sum (as defined in [BGR], 2.1.5) of $|\mathcal{R}|$ copies of $D(H, K)$. The induced topology on $D(G, K)$ will be called the *direct sum topology* with respect to the decomposition (20). Furthermore, whenever $D(G, K)$ carries an arbitrary norm $||.||$ such that $||\lambda|| = \max_{g \in \mathcal{R}} ||\lambda_g g||$ for all $\lambda = \sum_{g \in \mathcal{R}} \lambda_g g \in D(G, K), \ \lambda_g \in D(H, K)$ we will say that (20) is *orthogonal* with respect to $||.||$.

As a special case the direct sum decomposition can be used to endow the distribution algebra of an arbitrary compact locally L-analytic group G with a structure of K-Fréchet-Stein algebra coming from a uniform open normal subgroup ([ST5], Thm. 5.1 and its proof). Since we will refer to this several times in the following we briefly explain the details: choose an open normal uniform subgroup H_0 of G_0 (it always exists by [DDMS], Cor. 8.34). There is the usual family of norms $||.||_r,\ p^{-1} < r < 1,\ r \in p^{\mathbb{Q}}$ on $D(H_0, K)$. Pick such a norm $||.||_r$ on $D(H_0, K)$ and consider the direct sum decomposition

$$D(G_0, K) = \oplus_{\mathcal{R}} D(H_0, K)g \tag{21}$$

where g runs through a system $\mathcal{R}$ of representatives for the cosets in $H\backslash G$ containing 1. Define for each $||.||_r$ the maximum norm q_r on $D(G_0, K)$. Here, one has $||h||_r = 1$ for all $h \in H_0$. Indeed, $||h-1||_{1/p} < 1$ by [DDMS], Thm. 7.7 and since $||.||_r$ and $||.||_{1/p}$ induce the same topology on $\mathbb{Z}_p[[G]]$ (by [ST5], Sect. 4) one must have $||h-1||_r < 1$. Since $||.||_r$ is multiplicative ([ST5], Thm. 4.5) this shows that the maximum norm does not depend on the choice of $\mathcal{R}$. In this sense each of the norms $||.||_r$ is extended to $D(G_0, K)$ resulting in a family of submultiplicative norms $q_r,\ p^{-1} < r < 1$ in $p^{\mathbb{Q}}$, on $D(G_0, K)$ inducing a K-Fréchet-Stein structure. It is crucial here that H is normal in G and that the r-norms on $D(H_0, K)$ do not depend on the choice of an ordered basis with respect to the p-valuation on H_0 (see the discussion in [ST5] after Thm. 4.10). Finally, $D(G, K)$ is endowed with the quotient norms under the canonical quotient map $D(G_0, K) \to D(G, K)$.

The following two lemmata are simple "completed" versions of the decomposition (21) and are only stated for future reference:

Lemma 4.3 *Let G be a compact locally L-analytic group and H a normal open subgroup such that H_0 is uniform. Let $\mathcal{R}$ be a system of representatives for $H\backslash G$ containing 1 and $t := |\mathcal{R}|$. Fix a norm $||.||_r,\ p^{-1} < r < 1$ in $p^{\mathbb{Q}}$, on $D(H_0, K)$. Endow $D(G_0, K)$ with the maximum norm q_r. Write $D_r(G_0, K)$ resp. $D_r(H_0, K)$ for the completions. One has a direct sum decomposition as left $D_r(H_0, K)$-modules*

$$D_r(G_0, K) = \oplus_{g_i \in \mathcal{R}} D_r(H_0, K)g_i \tag{22}$$

where

1. *$g_i D_r(H_0, K) = D_r(H_0, K)g_i$ for all i,*
2. *for any $1 \leq i, j \leq t$ there is $1 \leq k \leq t$ such that $g_i g_j \in g_k D_r(H_0, K)$,*
3. *for any $1 \leq i \leq t$ there is $1 \leq l \leq t$ such that $g_i^{-1} \in g_l D_r(H_0, K)$.*

Proof: We have a direct sum decomposition

$$D(G_0, K) = \oplus_{g_i \in \mathcal{R}} D(H_0, K)g_i$$

as left $D(H_0, K)$-modules where $D(H_0, K)$ (instead of $D_r(H_0, K)$) satisfies the three listed properties with respect to the g_i. This follows simply from the properties of the underlying decomposition of $C^{an}(G_0, K)$ and the normality of H. Now by construction $(D(G_0, K), q_r)$ is isometrically isomorphic as a normed left $D(H_0, K)$-module to the normed direct sum $\oplus_{\mathcal{R}} D(H_0, K)$. Hence, passing to completions gives the direct sum decomposition of $D_r(G_0, K)$ where the basis $g_1 = 1, ..., g_t$ satisfies the required properties. $\square$

Lemma 4.4 *Adopt the assumptions of the last lemma. Moreover, write $q_{\bar{r}}$ resp. $\|.\|_{\bar{r}}$ for the quotient norm of q_r on $D(G, K)$ resp. $||.||_r$ on $D(H, K)$. Write $D_r(G, K)$ resp. $D_r(H, K)$ for the completions. There is a direct sum decomposition*

$$D_r(G, K) = \oplus_{g_i \in \mathcal{R}} D_r(H, K) g_i$$

as left $D_r(H, K)$-modules where

1. *$g_i D_r(H, K) = D_r(H, K) g_i$ for all i,*
2. *for any $1 \leq i, j \leq t$ there is $1 \leq k \leq t$ such that $g_i g_j \in g_k D_r(H, K)$,*
3. *for any $1 \leq i \leq t$ there is $1 \leq l \leq t$ such that $g_i^{-1} \in g_l D_r(H, K)$.*

Moreover, $q_{\bar{r}}$ equals the maximum norm coming from $\|.\|_{\bar{r}}$ on $D_r(H, K)$. In particular, $q_{\bar{r}}$ restricts to $\|.\|_{\bar{r}}$ on $D_r(H, K) \subseteq D_r(G, K)$.

Proof: Consider the decomposition (22) of the preceding lemma. Denote by $I(G_0, K)$ resp. $I(H_0, K)$ the kernel ideals of the quotient maps. It immediately follows from the nature of the quotient map and the decomposition that $I(H_0, K) = I(G_0, K) \cap D(H_0, K)$ and $I(G_0, K) = \oplus_i I(H_0, K)\, g_i$. By the construction of q_r on $D(G_0, K)$ as a maximum norm passing to closures in $D_r(G_0, K)$ resp. $D_r(H_0, K)$ yields $I_r(G_0, K) = \oplus_{g_i \in \mathcal{R}} I_r(H_0, K) g_i$. Then passing to quotients gives the desired decomposition since $D_r(G, K) = D_r(G_0, K)/I_r(G_0, K)$ (and similarly for H). Finally, for $\bar{\lambda} = \sum_i \bar{\lambda}_i g_i \in D_r(G, K)$ with $\bar{\lambda}_i = \lambda_i \bmod I_r(H_0, K) \in D_r(H_0, K)/I_r(H_0, K) = D_r(H, K)$ one has

$$\begin{aligned} q_{\bar{r}}(\bar{\lambda}) = \inf_{\mu \in I_r(G_0,K)} q_r(\lambda + \mu) &= \inf_{\mu \in I_r(G_0,K)} \max_i ||(\lambda + \mu)_i||_r \\ &= \max_i \inf_{\mu \in I_r(G_0,K)} ||\lambda_i + \mu_i||_r \\ &= \max_i \inf_{\mu \in I_r(H_0,K)} ||\lambda_i + \mu||_r \\ &= \max_i ||\bar{\lambda}_i||_{\bar{r}}. \end{aligned}$$

$\square$

4.3 Restriction of norms to lower p-series subalgebras

A result is proved concerning the restriction of a norm on the distribution algebra of a uniform group to the distribution algebra of a subgroup from its lower p-series.

To do this, let H_0 be an arbitrary uniform group of dimension $d :=\dim_{\mathbb{Q}_p} H_0$. For $m \in \mathbb{N}_0$ denote by $H_0^{(m)}$ the $(m+1)$-th term in its lower p-series. Thus $H_0^{(0)} = H_0,\ H_0^{(m)} = H_0^{p^m}$. For fixed $m \in \mathbb{N}_0$ the group $H_0^{(m)}$ is uniform of dimension d. Hence, its own lower p-series gives rise to an integrally valued p-valuation $\omega^{(m)}$ on $H_0^{(m)}$ by Prop. 2.1. According to the discussion after this proposition we obtain a family of norms $\|.\|_r,\ 0 < r < 1$ on $D(H_0^{(m)}, K)$ inducing a K-Fréchet structure. In order to distinguish these sets of norms on the algebras $D(H_0^{(m)}, K)$ for varying m we denote them by $\|.\|_r^{(m)},\ 0 < r < 1$. Abbreviate for simplicity $||.||_r := ||.||_r^{(0)}$ for all r.

Consider the two p-valuations $\omega^{(0)}$ resp. $\omega^{(m)}$ on H_0 resp. $H_0^{(m)}$. Suppose $h_1, ..., h_d$ is an ordered basis of the p-valued group $(H_0,\ \omega^{(0)})$. Because of $H_0^{(k)} = (H_0^{(m)})^{(k-m)}$ for all $k \geq m$ the restriction of $\omega^{(0)}$ to the subgroup $H_0^{(m)}$ differs, as a real valued function, from $\omega^{(m)}$ simply by translation by m. It follows that the elements $h_1^{p^m}, ..., h_d^{p^m}$ constitute an ordered basis of the p-valued group $(H_0^{(m)}, \omega^{(m)})$. (Remark: This follows also directly from Prop. 2.1 since the $h_1^{p^m}, ..., h_d^{p^m}$ are an ordered minimal set of topological generators of $H_0^{(m)}$.) This suggests that the families of norms $(||.||_r^{(m)})_r$ for different m can be brought into relation. Indeed, one has the

Proposition 4.5 *Fix $m \geq 1$ and $0 < r < 1$. Suppose $r^{\kappa(p^m-1)} > p^{-1}$. Then $\|.\|_r$ on $D(H_0, K)$ restricts to $\|.\|_{r'}^{(m)}$ on the subring $D(H_0^{(m)}, K)$ where $r' = r^{p^m}$.*

Proof: Abbreviate $R := D(H_0, K),\ R_r := D_r(H_0, K),\ R^{(m)} := D(H_0^{(m)}, K)$. Use induction on m.

1. Let $m = 1$. Assuming $r^{\kappa(p-1)} > p^{-1}$ we must show that $\|.\|_r$ on R restricts to $\|.\|_{r^p}^{(1)}$ on $R^{(1)}$. Let $h_1, ..., h_d$ be an ordered basis for the p-valued group H_0. Write $b_i := h_i - 1 \in R$ and $\mathbf{b}^\alpha = \prod_i b_i^{\alpha_i}$. Then according to 2.1 every $\lambda \in R$ has a unique convergent expansion

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ such that, for any $0 < s < 1$, the set $\{\|d_\alpha \mathbf{b}^\alpha\|_s\}_\alpha$ is bounded. Conversely, any such expansion is convergent in R. The norms $\|.\|_s$ on R for

$0 < s < 1$ are given by

$$\|\lambda\|_s = \sup_\alpha \|d_\alpha \mathbf{b}^\alpha\|_s = \sup_\alpha |d_\alpha| s^{\kappa|\alpha|}.$$

In particular, $\|b_i^{\alpha_i}\|_s = s^{\kappa\alpha_i}$.

Now analogous results hold on the level $m = 1$: As explained, the elements $h_1^p, ..., h_d^p$ constitute an ordered basis for the p-valued group $H_0^{(1)}$. Put $b'_i := h_i^p - 1$ and $\mathbf{b}'^\alpha := b_1'^{\alpha_1} \cdots b_d'^{\alpha_d}$. Then every $\lambda \in R^{(1)}$ has a unique convergent expansion

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}'^\alpha \tag{23}$$

with $d_\alpha \in K$ such that, for any $0 < s < 1$, the set $\{\|d_\alpha \mathbf{b}'^\alpha\|_s^{(1)}\}_\alpha$ is bounded. Conversely, any such expansion is convergent in $R^{(1)}$. The norms $\|.\|_s^{(1)}$ on $R^{(1)}$ for $0 < s < 1$ are given by

$$\|\lambda\|_s^{(1)} = \sup_\alpha \|d_\alpha \mathbf{b}'^\alpha\|_s^{(1)} = \sup_\alpha |d_\alpha| s^{\kappa|\alpha|}.$$

Hence, to verify that $\|.\|_r$ restricts to $\|.\|_{r^p}^{(1)}$ on $R^{(1)}$ one must show that

$$\|\lambda\|_r = \sup_\alpha |d_\alpha| r^{p\kappa|\alpha|} \tag{24}$$

for a $\lambda \in R^{(1)}$ given as in (23). This is done in two steps:

(i) First, compute the norm $\|\mathbf{b}'^\alpha\|_r$. Since $r^{\kappa(p-1)} > p^{-1}$ implies $r > p^{-1}$, the norm $\|.\|_r$ is multiplicative on R according to [ST5], Thm. 4.5. Hence, its restriction to $R^{(1)}$ is multiplicative and so $\|\mathbf{b}'^\alpha\|_r = \prod_i \|b'_i\|_r^{\alpha_i}$. Furthermore,

$$b'_i = h_i^p - 1 = (b_i + 1)^p - 1 = b_i^p + \sum_{k=1,...,p-1} \binom{p}{k} b_i^k.$$

Again by $r^{\kappa(p-1)} > p^{-1}$

$$\|b_i^p\|_r = r^{\kappa p} > p^{-1} r^\kappa \geq p^{-1} r^{\kappa k} = |\binom{p}{k}| r^{\kappa k} = \|\binom{p}{k} b_i^k\|_r$$

for all $k = 1, ..., p-1$ and so $\|b'_i\|_r = r^{\kappa p}$. Hence,

$$\|\mathbf{b}'^\alpha\|_r = r^{p\kappa|\alpha|}. \tag{25}$$

(ii) It will be shown that the family of pairwise different nonzero elements

$$\mathcal{T} := \{\mathbf{b}'^\alpha\}_{\alpha \in \mathbb{N}_0^d}$$

of $R^{(1)}$ satisfies the following condition: given a convergent series $\lambda \in R^{(1)}$ as in (23), i.e. $\lambda = \sum_\alpha d_\alpha \mathbf{b}'^\alpha$ one has

$$\|\lambda\|_r = \sup_\alpha \|d_\alpha \mathbf{b}'^\alpha\|_r.$$

Together with (25) this then yields the assertion (24).

The Fréchet topology on $R^{(1)}$ is stronger than the induced $||.||_r$-topology ([Ko], Prop. 1.1.3). This implies that the series $\lambda = \sum_\alpha d_\alpha \mathbf{b}'^\alpha$ is a convergent series in the normed K-Banach space $(R_r, ||.||_r)$. Thus, to prove the above condition it suffices to check that our family $\mathcal{T}$, indexed by $\mathbb{N}_0^d$, is an orthogonal family in $(R_r, ||.||_r)$. To see this put $e_i \in \mathbb{N}_0^d$ for the i-th unit vector and $pe_i := (0, ..., p, 0, ...)$ with i-th entry equal to p. The calculations above on the expansion

$$b'_i = \sum_k \binom{p}{k} b_i^k$$

for the element $b'_i = \mathbf{b}'^{e_i} \in \mathcal{T}$ show that an index S_{e_i} (in the notation of Lem. 4.1) equals necessarily pe_i and hence, is uniquely determined. By multiplicativity of $||.||_r$ and its ultrametric property it follows immediately that a product $\mathbf{b}'^\alpha = \prod_i b_i'^{\alpha_i} \in \mathcal{T}$ has also a unique index S_α, namely $\sum_i \alpha_i pe_i = (p\alpha_1, ..., p\alpha_n) = p\alpha$. But this means for our family $\mathcal{T}$ that all indices S_α are pairwise different and so Cor. 4.2 yields that $\mathcal{T}$ is an orthogonal family in $(R_r, ||.||_r)$.

2. Inductionstep. Let $m > 1$ and assume that the result holds true for all numbers strictly smaller than m. Furthermore, assume $r^{\kappa(p^m-1)} > p^{-1}$. We want to prove the result for m. Since $r^{\kappa(p^m-1)} > p^{-1}$ implies $r^{\kappa(p^{m-1}-1)} > p^{-1}$ the induction hypothesis shows that $\|.\|_r$ on R restricts to $\|.\|^{(m-1)}_{r^{p^{m-1}}}$ on $R^{(m-1)}$.

Now $H_0^{(m)}$ appears also as first step in the lower p-series of the uniform group $H_0^{(m-1)}$. Hence, the induction hypothesis applies to these two groups: for $0 < s < 1$, every $\|.\|^{(m-1)}_s$ on $R^{(m-1)}$ restricts to $\|.\|^{(m)}_{s^p}$ on $R^{(m)}$ as long as $s^{\kappa p} > s^\kappa p^{-1}$. We choose $s = r^{p^{m-1}}$. Then $s^{\kappa p} = r^{\kappa p^m} > r^\kappa p^{-1} > s^\kappa p^{-1}$ and so $\|.\|^{(m-1)}_{r^{p^{m-1}}}$ restricts to $\|.\|^{(m)}_{r^{p^m}}$ on $R^{(m)}$.

So $\|.\|_r$ restricts to $\|.\|^{(m)}_{r^{p^m}}$ on $R^{(m)}$ which completes the induction. □

4.4 Direct sums induced by lower p-series subgroups

In this subsection we will consider the distribution algebra of a uniform group and direct sum decompositions that arise from lower p-series subgroups. Thereby, Prop. 4.5 of the last subsection and its proof will play a crucial role.

Let H denote an arbitrary compact locally L-analytic group such that H_0 is uniform and has dimension $d := \dim_{\mathbb{Q}_p} H_0$. Given $m \in \mathbb{N}_0$ write as before $H_0^{(m)} := H_0^{p^m}$ for the $(m+1)$-th step in the lower p-series of H_0. It is again a uniform group. As explained previously, there is a family of multiplicative norms $||.||_r^{(m)}$, $p^{-1} < r < 1$, $r \in p^{\mathbb{Q}}$ on each $D(H_0^{(m)}, K)$ inducing a K-Fréchet-Stein structure. Note that $||h||_r^{(m)} = 1$ for all $h \in H_0^{(m)}$ and all r (by an argument above). Finally, abbreviate $||.||_r := ||.||_r^{(0)}$ for all r as before and put $\mathbb{N}_{0,<p}^d$ for the set of those indices of $\mathbb{N}_0^d$ with entries strictly smaller than p.

Lemma 4.6 *Assume* $1 > r^{\kappa(p^m-1)} > p^{-1}$ *for an* $r \in p^{\mathbb{Q}}$. *Fix a norm* $||.||_r$ *on* $D(H_0, K)$ *and restrict it to the subring* $D(H_0^{(m)}, K)$. *Fix a system of representatives* $\mathcal{R}$ *for* $H_0^{(m)} \backslash H_0$ *containing* 1. *Consider the usual decomposition of* $D(H_0, K)$ *as a left* $D(H_0^{(m)}, K)$*-module*

$$D(H_0, K) = \oplus_{h \in \mathcal{R}} D(H_0^{(m)}, K)\, h$$

and consider on $D(H_0, K)$ *the direct sum topology induced from* $(D(H_0^{(m)}, K), ||.||_r)$. *Then: The direct sum topology coincides on* $D(H_0, K)$ *with the* $||.||_r$*-topology. In particular, writing* $D_r(H_0, K)$ *for the completion of* $D_r(H_0, K)$ *along* $||.||_r$ *and* $D_{(r)}(H_0^{(m)}, K)$ *for the closure of* $D(H_0^{(m)}, K)$ *inside this completion one has*

$$D_r(H_0, K) = \oplus_{h \in \mathcal{R}} D_{(r)}(H_0^{(m)}, K)\, h. \tag{26}$$

Proof: Put $R^{(m)} := D(H_0^{(m)}, K)$, $R_r^{(m)} := D_{(r)}(H_0^{(m)}, K)$, $R := R^{(0)}$, $R_r := R_r^{(0)}$. First, let us remark that the two statements of the lemma are equivalent. Indeed, if the direct sum topology and the $||.||_r$-topology coincide on R the completion of R along $||.||_r$ equals the direct sum of the componentwise completions ([BGR], 2.1.5 Prop. 7). This gives (26). Conversely, if (26) is shown then $(R_r, ||.||_r)$ has a K-Banach space topology and is a finitely and freely generated topological module over the noetherian (since $R_r^{(m)} \subseteq R_r$ satisfies the assumptions of Prop. 5.3 below) K-Banach algebra $R_r^{(m)}$. Hence, by [ST5], Prop. 2.1 (i) the topology on R_r must be the direct sum topology.

Therefore, it suffices to prove (26). To do this we use induction on m.

1. Let $m = 1$ (for $m = 0$ there is nothing to show) and assume $r^{\kappa(p-1)} > p^{-1}$. We prove that $R_r = \oplus_{h \in \mathcal{R}} R_r^{(1)} h$.

Choose an ordered basis $h_1, ..., h_d$ of H_0 (as p-valued group) and put $b_i := h_i - 1,\ b'_i := h_i^p - 1$ for all i. As explained in the proof of Prop. 4.5 the elements $\mathbf{b}'^\alpha := b_1'^{\alpha_1} \cdots b_d'^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$ constitute an orthogonal basis of the K-Banach space $(R_r^{(1)}, ||.||_r)$.

By the discussion in [DDMS], 4.2 the elements $\mathbf{h}^\beta := h_1^{\beta_1} \cdots h_d^{\beta_d}$ with $\beta \in \mathbb{N}_{0,<p}^d$ constitute a system $\mathcal{R}'$ of representatives for $H_0^{(1)} \backslash H_0$. Consider the family of pairwise different nonzero elements

$$\mathcal{T} := \{\mathbf{b}'^\alpha\, \mathbf{b}^\beta\}_{(\alpha,\beta) \in \mathbb{N}_0^d \times \mathbb{N}_{0,<p}^d}$$

in R_r. We wish to apply Cor. 4.2 to obtain that $\mathcal{T}$ is an orthogonal basis for $(R_r, ||.||_r)$. First, observe that in the expansion

$$b'_i = h_i^p - 1 = (b_i + 1)^p - 1 = b_i^p + \sum_{k=1,...,p-1} \binom{p}{k} b_i^k$$

the inequality $r^{\kappa(p-1)} > p^{-1}$ implies (as in the proof of Prop. 4.5) that $||b'_i||_r > ||\binom{p}{k} b_i^k||_r$ for all $k = 1, ..., p-1$ and therefore $||b'_i||_r = ||b_i^p||_r$. Hence, in the notation of Cor. 4.2 the element $b'_i = \mathbf{b}'^{e_i}\mathbf{b}^0 \in \mathcal{T}$ has the uniquely defined index $S_{(e_i,0)} = p\, e_i$ (here, $e_i \in \mathbb{N}_0^d$ denotes the i-th unit vector). Then, by multiplicativity of $||.||_r$ and its ultrametric property an arbitrary element $\mathbf{b}'^\alpha \mathbf{b}^\beta \in \mathcal{T}$ has the unique index $p\alpha + \beta = (p\alpha_1 + \beta_1, ..., p\alpha_d + \beta_d)$. Hence, the set $\{S_{(\alpha,\beta)}\}_{(\alpha,\beta) \in \mathbb{N}_0^d \times \mathbb{N}_{0,<p}^d}$ consists of pairwise different elements. Moreover, to a given $\gamma \in \mathbb{N}_0^d$ there is clearly an element $\mathbf{b}'^\alpha \mathbf{b}^\beta$ of $\mathcal{T}$ with $S_{(\alpha,\beta)} = \gamma$. Thus, by Cor. 4.2 $\mathcal{T}$ is an orthogonal basis.

In particular, every element $\lambda \in R_r$ admits a unique convergent expansion

$$\lambda = \sum_{(\alpha,\beta) \in \mathbb{N}_0^d \times \mathbb{N}_{0,<p}^d} d_{\alpha,\beta} \mathbf{b}'^\alpha\, \mathbf{b}^\beta$$

with $d_{\alpha,\beta} \in K$. It follows by elementary properties of convergent series in complete ultrametrically normed rings (e.g. [DDMS], Prop. 6.10) that

$$\lambda = \sum_\beta \lambda_\beta \mathbf{b}^\beta$$

with $\lambda_\beta := \sum_\alpha d_{\alpha,\beta} \mathbf{b}'^\alpha$ a convergent series in $R_r^{(1)}$. The coefficients $d_{\alpha,\beta} \in K$ are uniquely determined. Moreover, by orthogonality of the family $\mathcal{T}$ in R_r with respect to $||.||_r$, orthogonality of the elements $\mathbf{b}'^\alpha$ in $R_r^{(1)}$ with respect to $||.||_r$ and by multiplicativity of $||.||_r$ one gets

$$||\lambda||_r = \sup_{\alpha,\beta} ||d_{\alpha,\beta} \mathbf{b}'^\alpha \mathbf{b}^\beta||_r = \sup_\beta \sup_\alpha ||d_{\alpha,\beta} \mathbf{b}'^\alpha||\, ||\mathbf{b}^\beta||_r = \sup_\beta ||\lambda_\beta \mathbf{b}^\beta||_r. \quad (27)$$

This proves

$$R_r = \oplus_{\beta \in \mathbb{N}^d_{0,<p}} R^{(1)}_r \, \mathbf{b}^\beta \tag{28}$$

as left $R^{(1)}_r$-modules and in an orthogonal manner with respect to $||.||_r$. Now use (28) to define an endomorphism of R_r as a left $R^{(1)}_r$-module via

$$\mathbf{b}^\beta \mapsto (b_1 + 1)^{\beta_1} \cdots (b_d + 1)^{\beta_d} = \mathbf{h}^\beta$$

for all $\beta \in \mathbb{N}^d_{0,<p}$. It is bijective since a direct computation by binomial expansions shows that the map $\mathbf{b}^\beta \mapsto (b_1 - 1)^{\beta_1} \cdots (b_d - 1)^{\beta_d}$ is a two-sided inverse. This yields $R_r = \oplus_{\beta \in \mathbb{N}^d_{0,<p}} R^{(1)}_r \mathbf{h}^\beta$ and since the occuring $\mathbf{h}^\beta$ constitute our system $\mathcal{R}'$ we obtain

$$R_r = \oplus_{h \in \mathcal{R}'} R^{(1)}_r h = \oplus_{h \in \mathcal{R}} R^{(1)}_r h.$$

2. Inductionstep. Let $m > 1$ and assume that the result holds true for all numbers strictly smaller than m. Furthermore, assume $r^{\kappa(p^m - 1)} > p^{-1}$. We prove that $R_r = \oplus_{h \in \mathcal{R}} R^{(m)}_r \, h$. By the induction hypothesis we can assume that

$$R_r = \oplus_{h \in \mathcal{R}'} R^{(m-1)}_r \, h \tag{29}$$

where $\mathcal{R}'$ is a system of representatives for $H^{(m-1)}_0 \backslash H_0$ containing 1. But $H^{(m)}_0$ appears also as first step in the lower p-series of $H^{(m-1)}_0$ and so the induction hypothesis applies to these two groups as well: if the index $p^{-1} < s < 1$ in $p^{\mathbb{Q}}$ satisfies $s^{\kappa p} > s^\kappa p^{-1}$ then one has

$$R^{(m-1)}_{[s]} = \oplus_{h \in \mathcal{R}''} R^{(m)}_{[s]} h$$

where $\mathcal{R}''$ is a system of representatives for $H^{(m)}_0 \backslash H^{(m-1)}_0$ containing 1, $R^{(m-1)}_{[s]}$ denotes the completion of $R^{(m-1)}$ along the norm $||.||^{(m-1)}_s$ and $R^{(m)}_{[s]}$ is the closure of $R^{(m)}$ inside this completion.

Now choose $s = r^{p^{m-1}}$. Then

$$s^{\kappa p} = r^{\kappa p^m} > r^\kappa p^{-1} > s^\kappa p^{-1}$$

and so

$$R^{(m-1)}_{[r^{p^{m-1}}]} = \oplus_{h \in \mathcal{R}''} R^{(m)}_{[r^{p^{m-1}}]} \, h. \tag{30}$$

But according to Prop. 4.5 our assumption $r^{\kappa(p^m - 1)} > p^{-1}$ implies that the norm $||.||_r$ on R restricts on $R^{(m-1)}$ to the norm $||.||^{(m-1)}_{r^{p^{m-1}}}$. Hence

$$R^{(m-1)}_{[r^{p^{m-1}}]} = R^{(m-1)}_r, \quad R^{(m)}_{[r^{p^{m-1}}]} = R^{(m)}_r \tag{31}$$

(recall that $R_r^{(m)}$ for varying m was defined to be the completion of $R^{(m)}$ via the restriction of $||.||_r$ from R to $R^{(m)}$). Therefore, (29) and (30) imply

$$R_r = \oplus_{h\in\mathcal{R}''\mathcal{R}'} R_r^{(m)}\, h$$

where $\mathcal{R}''\mathcal{R}' := \{h''h',\ h''\in\mathcal{R}'',\ h'\in\mathcal{R}'\}$ is a system of representatives for $H_0^{(m)}\backslash H_0$. Thus,

$$R_r = \oplus_{h\in\mathcal{R}} R_r^{(m)}\, h$$

which completes the induction. □

As a corollary the proof of the lemma yields a direct sum decomposition of $D_r(H_0, K)$ as a left $D_{(r)}(H_0^{(m)}, K)$-module (in the notation of the lemma) which is even orthogonal with respect to $||.||_r$:

Corollary 4.7 *Assume $1 > r^{\kappa(p^m-1)} > p^{-1}$ in $p^{\mathbb{Q}}$. Fix a norm $||.||_r$ on $D(H_0, K)$ and restrict it to the subring $D(H_0^{(m)}, K)$. Denote by $D_r(H_0, K)$ resp. $D_{(r)}(H_0^{(m)}, K)$ the completions. Let $h_1, ..., h_d$ be a basis for the p-valued group H_0. Consider the family $\mathcal{B}$ consisting of the following $p^{md} = (H_0 : H_0^{(m)})$ elements*

$$\mathbf{b}(m)^{\alpha(m)} \cdots \mathbf{b}(1)^{\alpha(1)} \in \mathbb{Z}[H]$$

where $\alpha(k) = (\alpha_1(k), ..., \alpha_d(k)) \in \mathbb{N}^d_{0,<p}$ and

$$\mathbf{b}(k)^{\alpha(k)} := (h_1^{p^{k-1}} - 1)^{\alpha_1(k)} \cdots (h_d^{p^{k-1}} - 1)^{\alpha_d(k)}$$

for all $k = 1, ..., m$. Then there is a direct decomposition

$$D_r(H_0, K) = \oplus_{b\in\mathcal{B}} D_{(r)}(H_0^{(m)}, K)b$$

which is orthogonal with respect to $||.||_r$:
given $\lambda = \sum_{b\in\mathcal{B}} \lambda_b b \in D_r(H_0, K),\ \lambda_b \in D_{(r)}(H_0^{(m)}, K)$ one has

$$||\lambda||_r = \max_{b\in\mathcal{B}} ||\lambda_b b||_r.$$

Proof: We use the same notation as in the preceding proof and again, induction on m.

1. The case $m = 1$ is explicitly contained in the above proof (see lines (27) and (28)).

2. Inductionstep. By the induction hypothesis we can assume that

$$R_r = \oplus_{b'\in\mathcal{B}'} R_r^{(m-1)} b' \tag{32}$$

in an orthogonal manner with respect to $||.||_r$ where

$$\mathcal{B}' = \{\mathbf{b}(m-1)^{\alpha(m-1)} \cdots \mathbf{b}(1)^{\alpha(1)},\ \alpha(k) \in \mathbb{N}^d_{0,<p}\}.$$

Moreover, the induction hypothesis applies (as explained in the preceding proof) to the groups $H_0^{(m)}$ and $H_0^{(m-1)}$ where now $h_1^{p^{m-1}}, ..., h_d^{p^{m-1}}$ is a basis for the p-valued group $H_0^{(m-1)}$ with respect to the p-valuation given by its own lower p-series: if the index $p^{-1} < s < 1$ in $p^{\mathbb{Q}}$ satisfies $s^{\kappa p} > s^{\kappa} p^{-1}$ then one has

$$R_{[s]}^{(m-1)} = \oplus_{b'' \in \mathcal{B}''} R_{[s]}^{(m)} b'' \tag{33}$$

in an orthogonal manner with respect to $||.||_s^{(m-1)}$ where now

$$\mathcal{B}'' = \{\mathbf{b}(m)^{\alpha(m)},\ \alpha(m) \in \mathbb{N}^d_{0,<p}\}.$$

Choosing $s := r^{p^{m-1}}$ as in the preceding proof, using (31) above and putting (32) and (33) together one gets the desired decomposition

$$R_r = \oplus_{b' \in \mathcal{B}'} (\oplus_{b'' \in \mathcal{B}''} R_r^{(m)} b'') b' = \oplus_{b \in \mathcal{B}} R_r^{(m)} b$$

since $\mathcal{B} = \{b''b',\ b'' \in \mathcal{B}'',\ b' \in \mathcal{B}'\}$. So it remains to check orthogonality with respect to $||.||_r$. Take any

$$\lambda = \sum_{b''b' \in \mathcal{B}''\mathcal{B}'} \lambda_{b''b'} b''b'$$

in R_r with $\lambda_{b''b'} \in R_r^{(m)}$. Put $\lambda_{b'} := \sum_{b'' \in \mathcal{B}''} \lambda_{b''b'} b'' \in R_r^{(m-1)}$. Then orthogonality of the decompositions (32) and (33), the fact that $||.||_{r^{p^{m-1}}}^{(m-1)} = ||.||_r$ on $R_r^{(m-1)}$ and multiplicativity of $||.||_r$ imply

$$\begin{aligned} ||\lambda||_r &= \max_{b'} ||\lambda_{b'} b'||_r = \max_{b'} ||\lambda_{b'}||_r ||b'||_r \\ &= \max_{b'} (\max_{b''} ||\lambda_{b''b'} b''||_r) \, ||b'||_r = \max_{b''b'} ||\lambda_{b''b'} b''b'||_r. \end{aligned}$$

$\square$

Using the above lemma we obtain a more general statement. Recall that the lower p-series of a (topologically) finitely generated pro-p group is a fundamental system of open neighbourhoods for 1.

Proposition 4.8 *Let N be an arbitrary open subgroup of H and let $\mathcal{R}$ be a system of representatives for $N\backslash H$ containing 1. Fix $m \geq 0$ such that $H_0^{(m)} \subseteq N$ and a radius $1 > r^{\kappa(p^m-1)} > p^{-1}$ in $p^{\mathbb{Q}}$. Endow $D(H_0, K)$ with the norm $||.||_r$ and restrict it to the subring $D(N_0, K)$. Then the direct sum topology on $D(H_0, K)$ with respect to the decomposition*

$$D(H_0, K) = \oplus_{h\in\mathcal{R}} D(N_0, K)h$$

coincides with the $||.||_r$-topology.

Proof: Let $\mathcal{R}'$ be a system of representatives for $H_0^{(m)}\backslash N$ containing 1. Then $\mathcal{R}'\mathcal{R} := \{h'h,\ h' \in \mathcal{R}',\ h \in \mathcal{R}\}$ is a system of representatives for $H_0^{(m)}\backslash H_0$ containing 1. By the preceding lemma we know that $(D(H_0, K),\ ||.||_r)$ carries the direct sum topology coming from

$$D(H_0, K) = \oplus_{h'h\in\mathcal{R}'\mathcal{R}} D(H_0^{(m)}, K)h'h.$$

So the induced $||.||_r$-topology on the direct summand

$$\oplus_{h'\in\mathcal{R}'} D(H_0^{(m)}, K)h' = D(N_0, K) \tag{34}$$

equals the direct sum topology induced from the decomposition (34). So all in all, the $||.||_r$-topology on $D(H_0, K)$ equals the direct sum topology coming from the decomposition $D(H_0, K) = \oplus_{h\in\mathcal{R}} D(N_0, K)h$. □

Recall that two norms on the same object are said to be *equivalent* if they induce the same topology.

Corollary 4.9 *Let N be an arbitrary open subgroup of H and let $\mathcal{R}$ be a system of representatives for $N\backslash H$ containing 1. Fix $m \geq 0$ such that $H_0^{(m)} \subseteq N$ and a radius $1 > r^{\kappa(p^m-1)} > p^{-1}$ in $p^{\mathbb{Q}}$. Fix $||.||_r$ on $D(H_0, K)$ and the quotient norm $\|.\|_{\bar{r}}$ on $D(H, K)$. Denote by res $||.||_r$ the restriction of $||.||_r$ to the subring $D(N_0, K)$ and by quot(res $||.||_r$) the quotient norm on $D(N, K)$. Let $D_r(H, K)$ resp. $D_{(r)}(N, K)$ denote the completions via these quotient norms. Then the restriction of $\|.\|_{\bar{r}}$ to $D(N, K)$ is equivalent to quot(res $||.||_r$) and one has as left $D_{(r)}(N, K)$-modules*

$$D_r(H, K) = \oplus_{h\in\mathcal{R}} D_{(r)}(N, K)h. \tag{35}$$

In particular, the $\|.\|_{\bar{r}}$-topology on $D_r(H, K)$ coincides with the direct sum topology induced by $(D_{(r)}(N, K),\ quot(res||.||_r))$ with respect to this decomposition.

Proof: One has $I(H_0, K) = \oplus_{h\in\mathcal{R}} I(N_0, K)h$ as a straightforward consequence of the nature of the quotient maps and the direct sum decomposition. By the preceding proposition $D(H_0, K) = \oplus_{h\in\mathcal{R}} D(N_0, K)h$ carries the direct sum topology induced by $(D(N_0, K), res\,||.||_r)$. Thus, we get a decomposition $I_r(H_0, K) = \oplus_{h\in\mathcal{R}} I_{(r)}(N_0, K)h$ of the closure $I_r(H_0, K)$ where $I_{(r)}(N_0, K)$ is the completion of $I(N_0, K)$ via $res\,||.||_r$. Then passing to quotients yields the decomposition (35) for $D_r(H, K)$ and the fact that the $\|.\|_{\bar{r}}$-topology on $D_r(H, K)$ equals the direct sum topology induced by $(D_{(r)}(N, K),\ quot(res\,||.||_r))$. This implies that $\|.\|_{\bar{r}}$ restricts on $D_{(r)}(N, K)$ to a norm which is equivalent to $quot(res\,||.||_r)$. □

Corollary 4.10 *Keep the assumptions of the preceding corollary but assume additionally that N is normal in H. Then the decomposition* (35) *satisfies*

1. $h_i D_{(r)}(N, K) = D_{(r)}(N, K)h_i$ *for all* i,
2. *for any* $1 \le i, j \le t$ *there is* $1 \le k \le t$ *such that* $h_i h_j \in h_k D_{(r)}(N, K)$,
3. *for any* $1 \le i \le t$ *there is* $1 \le l \le t$ *such that* $h_i^{-1} \in h_l D_{(r)}(N, K)$

where $\mathcal{R} =: \{h_1, ..., h_t\}$.

Proof: By the preceding proposition $D(H_0, K) = \oplus_{h\in\mathcal{R}} D(N_0, K)h$ carries the direct sum topology induced by $(D(N_0, K),\ res\,||.||_r)$. Hence, the proofs of Lem. 4.3 and Lem. 4.4 apply in an almost unchanged manner. □

Recall that the K-Fréchet-Stein structure of $D(G, K)$, G a compact locally L-analytic group is in fact *two-sided* meaning the noetherian property and the flatness property of the defining Banach algebras are satisfied from left and right ([ST5], Thm. 5.1 and proof).

Corollary 4.11 *Keep the assumptions of the preceding corollary. Consider the family $res\,\|.\|_{\bar{r}}$ on $D(N, K)$ where $1 > r^{\kappa(p^m-1)} > p^{-1}$ in $p^{\mathbb{Q}}$. It induces a two-sided K-Fréchet-Stein structure on $D(N, K)$.*

Proof: We have $m \ge 1$ and so $p^{-1} < r^{\kappa(p^m-1)}$ implies $p^{-1} < r$ and all restricted norms are multiplicative on $D(N, K)$. Since the inclusion $D(N, K) \subseteq D(H, K)$ is a topological embedding with respect to the Fréchet topologies ([Ko], Prop. 1.1.3) the topology induced on $D(N, K)$ by the given family of norms makes it a K-Fréchet algebra. Moreover, all completions $D_r(N, K) := D_{(r)}(N, K)$ are K-Banach algebras. Hence, by definition of a (two-sided) K-Fréchet-Stein algebra it remains to check that these are all noetherian rings and that transition between them is flat. By the above corollary there is a decomposition

$$D_r(H,K) = \oplus_{h\in\mathcal{R}} D_r(N,K)h = \oplus_{h\in\mathcal{R}} hD_r(N,K)$$

where $\mathcal{R}$ is a system of representatives for $N\backslash H$ containing 1. A chain of left ideals $(J_i)_i$ in $D_r(N,K)$ gives rise to a chain of left ideals $(D_r(H,K)J_i)_i$ in $D_r(H,K)$ which becomes stationary since $D_r(H,K)$ is left noetherian. Since $D_r(H,K)J_i \cap D_r(N,K) = J_i$ it follows that the ring $D_r(N,K)$ is left noetherian. The argument for right noetherian is similar. Now for flatness take two indices $r' < r$ and consider the isomorphism as bimodules

$$D_{r'}(H,K) \simeq D_{r'}(N,K) \otimes_{D_r(N,K)} D_r(H,K),$$

a direct consequence of the above decomposition (applied to r and r'). Abbreviate $E := D_{r'}(N,K)$, $A := D_r(N,K)$, $B := D_r(H,K)$. We want to prove that E is flat as a right A-module. Take an injection $M \hookrightarrow M'$ of left A-modules. Now E is flat as a right A-module and the A-bimodule B is right flat. According to [B-CA], I.2.7 Prop. 8 the right A-module $E \otimes_A B$ is flat whence

$$E \otimes_A B \otimes_A M \hookrightarrow E \otimes_A B \otimes_A M'.$$

But since B is in fact free as A-bimodule (of rank m, say) we have $E \otimes_A B \simeq E^m$ as right A-modules and thus one finally gets $E \otimes_A M \hookrightarrow E \otimes_A M'$.

The argument for left flatness is similar using the decomposition

$$D_{r'}(H,K) \simeq D_r(H,K) \otimes_{D_r(N,K)} D_{r'}(N,K)$$

as bimodules. □

Remark: In the next section we will study the behaviour of certain homological properties under ring extensions

$$D_{(r)}(N,K) \subseteq D_r(H,K)$$

as above (cf. Prop. 5.5).

5 Auslander regularity

This section is devoted to the construction of a K-Fréchet-Stein structure for $D(G,K)$ that is *regular* in the following sense: all defining Banach algebras are Auslander regular rings (see definition below) whose global dimension is bounded above by $\dim_L G$. In [ST5] the authors prove that such a structure exists if $L = \mathbb{Q}_p$. In the following we show that this generalizes to the case of a finite extension $L/\mathbb{Q}_p$. The main work will be to prove that any open subgroup H of G such that H_0 is uniform and satisfies condition (L) (formulated in 2.2) has such a structure. The latter will come as usual as a quotient structure from $D(H_0,K)$ via $D(H_0,K) \to D(H,K)$.

5.1 Some results on Auslander regularity

The notion of Auslander regularity is briefly recalled (cf. [LVO], Chap. III). A result is proved about how this property behaves under suitable ring extensions.

Let R be a left and right noetherian ring. For any left R-module N the *grade number* $j_R(N)$ is the unique smallest integer l such that $\mathrm{Ext}^l_R(N,R) \neq 0$ ([LVO], III.2.2.1). If such an integer does not exist one writes $j_R(N) = \infty$. For finitely generated $N \neq 0$ the grade is bounded above by the projective dimension of N. For right modules one has the analogous definition.

Now a left R-module N is said to satisfy the *Auslander condition* (AC) if for any $l \geq 0$ and any right R-submodule $L \subseteq \mathrm{Ext}^l_R(N,R)$ one has $j_R(L) \geq l$.

A left and right noetherian ring R of finite global dimension is called *Auslander regular* if every finitely generated left or right R-module N satisfies the Auslander condition (AC) ([LVO], III.2.1.7).

Proposition 5.1 *Any commutative noetherian ring with finite global dimension is Auslander regular.*

Proof: This is [LVO], III.2.4.3. □

Proposition 5.2 *Let R be a complete filtered ring (for a brief account on filtered rings, see* [ST5]*, Sect. 1). Assume that the graded ring is Auslander regular of global dimension l. Then R is Auslander regular of global dimension $\leq l$.*

Proof: This is [LVO], II.3.1.4 and III.2.2.5 together with the fact that a complete filtered ring with noetherian graded ring is a Zariski ring ([LVO], II.2.2.1). □

Proposition 5.3 *Let $R_0 \subseteq R_1$ be an extension of unital rings. Suppose there are units $b_1 = 1, b_2, ..., b_t \in R_1^\times$ which form a basis of R_1 as (left and right) R_0-module and which satisfy:*

1. $b_i R_0 = R_0 b_i$ for any $1 \leq i \leq t$,

2. for any $1 \leq i, j \leq t$ there is $1 \leq k \leq t$ s.t. $b_i b_j \in b_k R_0$,

3. for any $1 \leq i \leq t$ there is $1 \leq l \leq t$ s.t. $b_i^{-1} \in b_l R_0$.

Suppose t is invertible in R_0.

Then: R_0 is noetherian if and only if R_1 is noetherian. In this case both rings have the same global dimension. R_0 is an Auslander regular ring if and only if this holds true for R_1.

Proof: Clearly, if R_0 is (left and right) noetherian so is R_1. Conversely, let R_1 be noetherian and consider an ascending chain of left ideals (J_i) in R_0. The chain of left ideals $(R_1 J_i)$ in R_1 becomes stationary. Now $R_1 = \oplus_i b_i R_0$ implies $R_1 J_i \cap R_0 = J_i$ and so the chain (J_i) becomes stationary. The same works with right ideals thus proving that R_1 is noetherian. In this case [ST5], Lem. 8.8 (note, that in [loc.cit.] the assumption that t is invertible is crucial but not explicitly stated) yields that R_0 and R_1 have the same global dimension.

Now assume that R_0 is Auslander regular. Let N be a finitely generated left or right R_1-module. By [loc.cit.] we have

$$\mathrm{Ext}^*_{R_1}(N, R_1) \simeq \mathrm{Ext}^*_{R_0}(N, R_0) \tag{36}$$

as abelian groups. Now N has finite projective dimension since gld R_1=gld R_0 is finite. Next, let $L \subseteq \mathrm{Ext}^l_{R_1}(N, R_1)$ be any R_1-submodule. Since R_1 is noetherian and N is finitely generated $\mathrm{Ext}^l_{R_1}(N, R_1)$ is finitely generated as R_1-module. Hence, so is L. Consider L as R_0-module. Then it is finitely generated and so from $L \subseteq \mathrm{Ext}^l_{R_0}(N, R_0)$ we deduce by (AC) for the finitely generated R_0-module N that $j_{R_0}(L) \geq l$. But this implies $j_{R_1}(L) \geq l$ by (36). Hence, N satisfies (AC) and R_1 is Auslander regular.

Conversely, suppose that R_1 is Auslander regular. Let N be a finitely generated left R_0-module. Then it has finite projective dimension and we prove (AC): let $L \subseteq \mathrm{Ext}^l_{R_0}(N, R_0)$ be any right R_0-module. It is finitely generated by the same argument as above. Put $N_1 := R_1 \otimes_{R_0} N$. Then

$$L \otimes_{R_0} R_1 \subseteq \mathrm{Ext}^l_{R_0}(N, R_0) \otimes_{R_0} R_1 = \mathrm{Ext}^l_{R_1}(N_1, R_1)$$

where the first inclusion is flatness of R_1 over R_0 and the last equality (of right R_1-modules) follows from Lemma 5.4 below. By (AC) for N_1 we have $j_{R_1}(L \otimes_{R_0} R_1) \geq l$ and so $\mathrm{Ext}^k_{R_1}(L \otimes_{R_0} R_1, R_1) = 0$ for all $k < l$. By the same proof as for Lem. 5.4 (replace "left" by "right") one has

$$\mathrm{Ext}^k_{R_1}(L \otimes_{R_0} R_1, R_1) = R_1 \otimes_{R_0} \mathrm{Ext}^k_{R_0}(L, R_0)$$

as left R_1-modules and so $\mathrm{Ext}^k_{R_0}(L, R_0) = 0$ for all $k < l$ by faithful flatness of R_1. This implies by definition of the grade number $j_{R_0}(L) \geq l$ and so the left R_0-module N satisfies (AC). The proof for right modules being the same this shows that R_0 is Auslander regular. □

Lemma 5.4 *Let $R_0 \subseteq R_1$ be an extension of unital noetherian rings where R_1 is flat as left and right R_0-module. Let N be a finitely generated left R_0-module. Put $N_1 = R_1 \otimes_{R_0} N$. Then*

$$\mathrm{Ext}^*_{R_0}(N, R_0) \otimes_{R_0} R_1 = \mathrm{Ext}^*_{R_1}(N_1, R_1)$$

as right R_1-modules.

Proof: This is a standard argument: choose a projective resolution

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

of N by finitely generated free left R_0-modules. Then applying the functor $R_1 \otimes_{R_0} (.)$ to this sequence gives a projective resolution of N_1. Trivially, one has canonical isomorphisms of right R_1-modules

$$\mathrm{Hom}_{R_1}(R_1 \otimes_{R_0} P_i, R_1) \simeq \mathrm{Hom}_{R_0}(P_i, R_0) \otimes_{R_0} R_1$$

for each i which induce an isomorphism between the complexes $\mathrm{Hom}_{R_1}(R_1 \otimes_{R_0} P_\bullet, R_1)$ and $\mathrm{Hom}_{R_0}(P_\bullet, R_0) \otimes_{R_0} R_1$. Hence, the modules $\mathrm{Ext}^*_{R_1}(N_1, R_1)$ can be obtained by taking cohomology in the latter complex. But by flatness of R_1 taking cohomology commutes with the functor $(.) \otimes_{R_0} R_1$ on the complex $\mathrm{Hom}_{R_0}(P_\bullet, R_0)$ which gives the claim. □

Let H be a compact locally L-analytic group such that H_0 is uniform and satisfies condition (L). Let $\|.\|_{\bar{r}},\ p^{-1} < r < 1,\ r \in p^{\mathbb{Q}}$ be the set of quotient norms on $D(H, K)$ inducing its usual K-Fréchet-Stein structure.

Proposition 5.5 *Let N be an open normal subgroup of H. Let $\mathcal{R}$ be a system of representatives for $N\backslash H$ containing 1. Fix $m \in \mathbb{N}_0$ such that $H_0^{(m)} \subseteq N$ and a norm $\|.\|_{\bar{r}}$ on $D(H,K)$ with $1 > r^{\kappa(p^m-1)} > p^{-1}$. Denote by $D_{(r)}(N,K)$ as before the closure of $D(N,K)$ inside $D_r(H,K)$. The ring extension*

$$D_{(r)}(N,K) \subseteq D_r(H,K)$$

satisfies the assumptions of Prop. 5.3 with $\mathcal{R}$ as a module basis. In particular, both rings are noetherian, have the same global dimension and $D_r(H,K)$ is an Auslander regular ring if and only if this is true for $D_{(r)}(N,K)$.

Proof: This follows from Cor. 4.10 and Prop. 5.3. □

5.2 A regular Fréchet-Stein structure for $D(G,K)$

We will prove that for any open subgroup H of G such that H_0 is uniform satisfying condition (L) the algebra $D(H,K)$ has a regular K-Fréchet-Stein structure. Choosing, in addition, H to be normal in G (such an H exists by Cor. 2.8) this generalizes to $D(G,K)$.

Let H be an open subgroup of G such that H_0 is uniform and satisfies condition (L). For any $m \geq 0$ let $H^{(m)}$ be the $(m+1)$-th step in its lower p-series. Then $H_0^{(m)}$ is uniform satisfying (L) by Cor. 2.9. Furthermore, there is the usual family of norms $||.||_r^{(m)}$, $p^{-1} < r < 1$, $r \in p^{\mathbb{Q}}$ on $D(H_0^{(m)},K)$ inducing a K-Fréchet-Stein structure. As previously explained this family is induced by the p-valuation on $H_0^{(m)}$ coming from its own lower p-series. Fix bases $\mathfrak{x}_1,...,\mathfrak{x}_d$ resp. $v_1 = 1,...,v_n$ of $\mathfrak{g}_L$ over L resp. $\mathfrak{o}$ over $\mathbb{Z}_p$ that realize condition (L) for $H_0^{(m)}$. Form the $nd-d$ elements F_{ij} as usual, i.e. $F_{ij} = \partial_{ij} - v_i\partial_{1j}$, $i \geq 2$, $j \geq 1$ with $\partial_{ij} = v_i\mathfrak{x}_j \in D_r(H_0^{(m)},K)$. Using canonical expansions for $D(H_0^{(m)},K)$ via the global chart

$$H_0^{(m)} \xrightarrow{\theta_{\mathbb{Q}_p}^{-1}} \oplus_j \oplus_i \mathbb{Z}_p v_i\mathfrak{x}_j \longrightarrow \mathbb{Z}_p^{nd}$$

one obtains with $h_{ij} := \theta_{\mathbb{Q}_p}(v_i\mathfrak{x}_j) \in H_0^{(m)}$ and $b_{ij} := h_{ij} - 1 \in D(H_0^{(m)},K)$ that $\partial_{ij} = \log(1+b_{ij})$. All this was already explained in the previous subsections. Given a norm $||.||_r^{(m)}$ on $D(H_0^{(m)},K)$ denote by $\|.\|_{\bar{r}}^{(m)}$ as usual the corresponding quotient norm on $D(H^{(m)},K)$. As usual we abbreviate $||.||_r := ||.||_r^{(0)}$ and $\|.\|_{\bar{r}} := \|.\|_{\bar{r}}^{(0)}$ for all r.

Proposition 5.6 *Fix an index $p^{-1} < r < 1$ in $p^{\mathbb{Q}}$ with $r^\kappa < p^{-\frac{1}{p-1}}$ and the quotient norm $\|.\|_{\bar{r}}^{(m)}$ on $D(H^{(m)}, K)$. Denote by $gr^{\cdot}_r D_r(H^{(m)}, K)$ the graded ring associated to the $\|.\|_{\bar{r}}^{(m)}$-filtration on the completion $D_r(H^{(m)}, K)$. Then $gr^{\cdot}_r D_r(H^{(m)}, K)$ is a polynomial ring over $gr^{\cdot} K$ in d variables and the norm $\|.\|_{\bar{r}}^{(m)}$ on $D_r(H^{(m)}, K)$ is multiplicative. Furthermore, $D_r(H^{(m)}, K)$ is an integral domain, Auslander regular and of global dimension $\leq d+1$.*

Proof: According to Prop. 3.18 it remains to prove the Auslander regularity and the bound on the global dimension. Note that by [MCR], Thm. 7.5.3 the ring $gr^{\cdot} K = k[\epsilon_0, \epsilon_0^{-1}]$ resp. $gr^{\cdot}_r D_r(H^{(m)}, K) = (gr^{\cdot} K)[X_{11}, ..., X_{1d}]$ has global dimension 1 resp. $d+1$. Thus, $gr^{\cdot}_r D_r(H^{(m)}, K)$ is a commutative noetherian ring of global dimension $d+1$. Combining Prop. 5.1 with Prop. 5.2 yields that the ring $D_r(H^{(m)}, K)$ is Auslander regular with global dimension $\leq d+1$. $\square$

Corollary 5.7 *Keeping the assumptions of the preceding proposition one has the sharper bound*

$$\text{gld } D_r(H^{(m)}, K) \leq d.$$

Proof: This is a generalization of the argument given in [ST5], (proof of) Thm. 4.9: first, we may pass to a finite extension of K if necessary. Indeed, if K'/K is finite then it follows directly from the constructions that $D_r(H^{(m)}, K')$ is free and finitely generated as a (left or right) module over $D_r(H^{(m)}, K)$. Since we already know that gld $D_r(H^{(m)}, K) < \infty$ we obtain gld $D_r(H^{(m)}, K) \leq$ gld $D_r(H^{(m)}, K')$ by [MCR], Thm. 7.2.6.

According to this remark we may assume that the ramification index e of K satisfies $p^{-l/e} = r^\kappa$ (recall that $r \in p^{\mathbb{Q}}$) with suitable $l \in \mathbb{N}$. It follows that homogeneous components in the graded ring $gr^{\cdot}_r D_r(H^{(m)}, K)$ are nonzero if and only if their degree lies in $1/e \cdot \mathbb{Z}$. Denote by $F^0_r D_r(H^{(m)}, K)$ the subring of $D_r(H^{(m)}, K)$ equal to the zero-term in the $\|.\|_{\bar{r}}^{(m)}$-filtration of $D_r(H^{(m)}, K)$. We prove

$$\text{gld } F^0_r D_r(H^{(m)}, K) \leq d+1. \tag{37}$$

From this the claim follows since gld $D_r(H^{(m)}, K) <$ gld $F^0_r D_r(H^{(m)}, K)$ via the same argument as in the proof of [ST5], Thm. 8.9.

So let us prove the inequality (37). By Prop. 3.20 any element $\lambda \in D_r(H^{(m)}, K)$ admits a unique convergent expansion

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ in the monomials $\mathbf{b}^\alpha = b_1^{\alpha_1} \cdots b_d^{\alpha_d}$. Here, b_j is the image of b_{1j} under the quotient map $D_r(H_0^{(m)}, K) \to D_r(H^{(m)}, K)$. In this situation the norm is computed via

$$||\lambda||_{\bar{r}}^{(m)} = \sup_\alpha |d_\alpha| r^{\kappa|\alpha|}.$$

Now let π be a prime element of K and $s \in \mathbb{N}_0$. Then $r^{\kappa|\alpha|} = |\pi|^{l\,|\alpha|}$ for all $\alpha \in \mathbb{N}_0^d$ and so for fixed $e_\alpha \in K$ the condition

$$||e_\alpha (b_1/\pi^l)^{\alpha_1} ... (b_d/\pi^l)^{\alpha_d}||_{\bar{r}}^{(m)} \leq p^{-s/e}$$

is equivalent to $|e_\alpha| \leq |\pi|^s$. It follows that the set $F_r^{s/e} F_r^0 D_r(H^{(m)}, K)$ consists precisely of all series

$$\sum_\alpha e_\alpha (b_1/\pi^l)^{\alpha_1} ... (b_d/\pi^l)^{\alpha_d}$$

with $|e_\alpha| \leq |\pi|^s$ and $|e_\alpha| \to 0$ for $|\alpha| \to \infty$. By Cor. 3.21 the isomorphism

$$gr_r^\cdot\, D_r(H^{(m)}, K) \xrightarrow{\sim} (gr^\cdot K)[X_{11}, ..., X_{1d}],$$

maps $\sigma(b_j) \mapsto X_{1j}$. Thus, restricting it to the homogeneous component $gr_r^{s/e} F_r^0 D_r(H^{(m)}, K)$ of the subring $gr_r^\cdot\, F_r^0 D_r(H^{(m)}, K) \subseteq gr_r^\cdot\, D_r(H^{(m)}, K)$ yields an isomorphism (of additive groups)

$$gr_r^{s/e} F_r^0 D_r(H^{(m)}, K) \simeq \epsilon_0^s \cdot k[u_1, ..., u_d]$$

where $\epsilon_0 = \sigma(\pi),\ u_j := \sigma(b_j/\pi^l) = \epsilon_0^{-l} X_{1j}$. Letting s vary one arrives at the ring isomorphism

$$gr_r^\cdot\, F_r^0 D_r(H^{(m)}, K) \simeq k[\epsilon_0, u_1, ..., u_d].$$

The right-hand side is a commutative noetherian ring of global dimension $d+1$ ([MCR], Thm. 7.5.3). Since $F_r^0 D_r(H^{(m)}, K)$ is closed in $D_r(H^{(m)}, K)$ it follows by Prop. 5.2 that it has global dimension $\leq d+1$. $\square$

Now make the following definition: fix once and for all a radius $\delta \in p^{\mathbb{Q}}$ with $p^{-1} < \delta < 1$ such that $\delta^\kappa < p^{-\frac{1}{p-1}}$. (For $p = 2$ we obviously have $p^{-1} = p^{-\frac{1}{p-1}}$ but at the same time $\kappa = 2$). Consider the sequence

$$\delta_m := \delta^{1/p^m}$$

of its positive real p-power roots. Clearly $p^{-1} < \delta_m < 1$ and $\delta_m \uparrow 1$ in $p^{\mathbb{Q}}$ for $m \to \infty$. Let $S(\delta)$ be the set of all $\delta_m,\ m \geq 1$.

Hence, given $r \in S(\delta)$ we may choose $m \in \mathbb{N}_0$ such that

$$r^{p^{m+\kappa-1}} = \delta$$

and consider the algebra $D_\delta(H^{(m+\kappa-1)}, K)$. The latter denotes as usual the completion of the algebra $D(H^{(m+\kappa-1)}, K)$ via the quotient norm $||.||_{\bar{\delta}}^{(m+\kappa-1)}$ out of the canonical family of quotient norms $||.||_{\bar{s}}^{(m+\kappa-1)}$, $p^{-1} < s < 1$, $s \in p^{\mathbb{Q}}$ on $D(H^{(m+\kappa-1)}, K)$.

Lemma 5.8 *Let H be a locally L-analytic group such that H_0 is uniform and satisfies condition* (L). *Fix $r \in S(\delta)$. Let $m \in \mathbb{N}_0$ such that $r^{p^{m+\kappa-1}} = \delta$. Let $\mathcal{R}$ be a system of representatives for $H^{(m+\kappa-1)} \backslash H$ containing* 1. *The restriction $res\,\|.\|_{\bar{r}}$ of $\|.\|_{\bar{r}}$ from $D(H, K)$ to $D(H^{(m+\kappa-1)}, K)$ is equivalent to the norm $||.||_{\bar{\delta}}^{(m+\kappa-1)}$. The ring extension*

$$D_\delta(H^{(m+\kappa-1)}, K) \subseteq D_r(H, K)$$

satisfies the assumptions of Prop. 5.3. In particular, both rings have the same global dimension and $D_r(H, K)$ is Auslander regular if and only if this is true for $D_\delta(H^{(m+\kappa-1)}, K)$.

Proof: We distinguish two cases:

1. $p \neq 2$.

Then $\kappa = 1$ and $r^{p^m} = \delta$. Now $p^{-1} < \delta = r^{\kappa p^m}$ implies $p^{-1} < r^{\kappa(p^m-1)}$. Thus Cor. 4.9 yields that $res\,\|.\|_{\bar{r}}$ is equivalent to $quot(res\,||.||_r)$ on the subring $D(H^{(m)}, K) \subseteq D(H, K)$ where as usual $res\,||.||_r$ denotes the restriction of $||.||_r$ to the subring $D(H_0^{(m)}, K)$ and $quot(res\,||.||_r)$ denotes the quotient norm of this restriction. But according to Prop. 4.5 the two norms $res\,||.||_r$ and $||.||_{r^{p^m}}^{(m)}$ coincide on $D(H_0^{(m)}, K)$. Thus $res\,\|.\|_{\bar{r}}$ is equivalent to $quot(||.||_{r^{p^m}}^{(m)}) = ||.||_{\bar{\delta}}^{(m+\kappa-1)}$ on $D(H^{(m)}, K) = D(H^{(m+\kappa-1)}, K)$. We thus have a ring extension

$$D_\delta(H^{(m+\kappa-1)}, K) \subseteq D_r(H, K)$$

and, again by $p^{-1} < r^{\kappa(p^m-1)}$, Prop. 5.5 yields all claimed properties of it.

2. $p = 2$.

Then $\kappa = 2$ and $r^{p^{m+1}} = \delta$. Then $p^{-1} < \delta = r^{\kappa p^m}$ implies $p^{-1} < r^{\kappa(p^m-1)}$ and so, as for $p \neq 2$, the norm $res\,\|.\|_{\bar{r}}$ is equivalent to $quot(||.||_{r^{p^m}}^{(m)})$ on $D(H^{(m)}, K)$.

Now put $s := r^{p^m}$. We wish to apply Prop. 4.5 and Cor. 4.9 a second time, namely to $p^{-1} < s < 1$ in $p^{\mathbb{Q}}$, the norm $||.||_s^{(m)}$ on $D(H_0^{(m)}, K)$ and the subgroup $H_0^{(m+1)}$. This is possible since

$$s^{\kappa p} = r^{p^{m+2}} = r^{2\,p^{m+1}} = \delta^2 > p^{-1}\delta = p^{-1} r^{p^{m+1}} = p^{-1} r^{\kappa\, p^m} = p^{-1} s^\kappa$$

i.e. $s^{\kappa(p-1)} > p^{-1}$. By Prop. 4.5 the restriction of $||.||^{(m)}_{r^{p^m}}$ and the norm $||.||^{(m+1)}_{r^{p^{m+1}}}$ coincide on $D(H_0^{(m+1)}, K)$. Thus, the restriction of $quot(||.||^{(m)}_{r^{p^m}})$ to $D(H^{(m+1)}, K)$ is equivalent to $quot(||.||^{(m+1)}_{r^{p^{m+1}}})$ according to Cor. 4.9. All in all $res\,\|.\|_{\bar{r}}$ is equivalent to $quot(||.||^{(m+1)}_{r^{p^{m+1}}}) = ||.||^{(m+\kappa-1)}_{\bar{\delta}}$ on $D(H^{(m+1)}, K) = D(H^{(m+\kappa-1)}, K)$. We thus have a ring extension

$$D_\delta(H^{(m+\kappa-1)}, K) \subseteq D_r(H, K)$$

and, applying Prop. 5.5 in the same manner twice, it has the claimed properties. □

Theorem 5.9 *Let H be a locally L-analytic group such that H_0 is uniform and satisfies condition* (L). *Consider the usual family of quotient norms $\|.\|_{\bar{r}},\ p^{-1} < r < 1,\ r \in p^{\mathbb{Q}}$ on $D(H, K)$. If the index r lies in the set $S(\delta)$ the completed ring $D_r(H, K)$ is Auslander regular and of global dimension $\leq d$.*

Proof: Let $r \in S(\delta)$ and $m \in \mathbb{N}_0$ such that $r^{p^{m+\kappa-1}} = \delta$. By the above lemma it suffices to show that $D_\delta(H^{(m+\kappa-1)}, K)$ is an Auslander regular ring of global dimension $\leq d$. But since $\delta^\kappa < p^{-\frac{1}{p-1}}$ this follows from Prop. 5.6 and Cor. 5.7. □

Again, recall that any compact locally L-analytic group G contains an open normal subgroup H such that H_0 is uniform and satisfies condition (L) (Cor. 2.8).

Theorem 5.10 *Let G be a compact locally L-analytic group of dimension d. Choose an open normal subgroup H such that H_0 is uniform satisfying condition* (L). *Consider the set of norms $||.||_r,\ r \in S(\delta)$ on $D(H_0, K)$. Endow $D(G_0, K)$ with the maximum norms and $D(G, K)$ with the quotient norms. This gives a K-Fréchet-Stein structure on $D(G, K)$. The arising K-Banach algebras $D_r(G, K)$ are Auslander regular rings of global dimension $\leq d$.*

Proof: Let $quot(||.||_r),\ r \in S(\delta)$ be the family of quotient norms on $D(H, K)$ via the quotient map $D(H_0, K) \to D(H, K)$ and let $D_r(H, K)$ be the completions. Choose a system $\mathcal{R}$ of representatives $g_1 = 1, ..., g_l$ of $H\backslash G$ and use the decomposition $D(G_0, K) = \oplus_i D(H_0, K) g_i$ to define on $D(G_0, K)$ for each $r \in S(\delta)$ the maximum norm. Put on $D(G, K)$ the quotient norms $\|.\|_{\bar{r}}$

and let $D_r(G,K)$ be the completions. According to Lem. 4.4 there is a direct sum decomposition

$$D_r(G,K) = \oplus_{g\in\mathcal{R}} D_r(H,K)g$$

satisfying the assumptions of Prop. 5.3. Hence, the ring $D_r(G,K)$ is Auslander regular of global dimension $\leq d$ if and only if this is true for $D_r(H,K)$. Hence, the preceding theorem completes the proof. □

Since the Fréchet-Stein structure on $D(G,K)$ exhibited here arises in the same manner (modulo the cofinal index set $S(\delta)$ and the additional condition on the inducing uniform subgroup H) as in [ST5] we will in the following tacitly assume that $D(G,K)$ is always endowed with this *regular* structure. It is clearly two-sided.

6 Dimension theory

In this section we explain how the dimension theory on the category of coadmissible $D(G,K)$-modules as developed by Schneider/Teitelbaum in [ST5], Sect. 8 for the base field $\mathbb{Q}_p$ extends to finite extensions $L/\mathbb{Q}_p$.

Recall that starting from an abstract (left) K-Fréchet-Stein algebra $A = (A_{q_n})$ with defining Banach algebras A_{q_n} one may construct the category $\mathcal{C}_A$ of coadmissible A-modules. It is a full subcategory of the (left) A-modules and enjoys good algebraic properties (abelian, closed under extensions etc.). The main motivation of studying $\mathcal{C}_A$ comes from the special case $A := D(G,K)$ where G is a compact locally L-analytic group. In this case $\mathcal{C}_G := \mathcal{C}_{D(G,K)}$ is anti-equivalent to the category $\mathrm{Rep}_K^a(G)$ of admissible locally analytic G-representations via the functor "passing to the strong dual" ([ST5], Thm. 6.3) and thus serves as an algebraization of the locally analytic representation theory of G.

To obtain more structure on $\mathcal{C}_A$ the authors of [ST5] introduce the following hypothesis:

(DIM) There is an integer $d \geq 0$ such that each A_{q_n} is an Auslander regular ring of global dimension $\leq d$.

It is then shown that under the assumption that the Fréchet-Stein structure of A is two-sided and satisfies (DIM) the category $\mathcal{C}_A$ admits a well-behaved codimension theory. Here, the grade number j_A (as defined in section 5) acts as a codimension function. Furthermore, any $M \in \mathcal{C}_A$ comes equipped with a filtration

$$M = \Delta^0(M) \supseteq \Delta^1(M) \supseteq ... \supseteq \Delta^{d+1}(M) = 0$$

by coadmissible submodules which is characterized by the property (*) that a coadmissible submodule $N \subseteq M$ has grade number $j_A(N) \geq l$ if and only if $N \subseteq \Delta^l(M)$ (see [loc.cit.], Prop. 8.7 for further properties).

It is one of the main results of [loc.cit.] that in case of a compact locally $\mathbb{Q}_p$-analytic group G the two-sided Fréchet-Stein algebra $D(G, K)$ satisfies (DIM) with $d = \dim_{\mathbb{Q}_p} G$ ([loc.cit.], Thm. 8.9). The following generalization of this result is just a reformulation of Thm. 5.10 above.

Theorem 6.1 *Given any compact locally L-analytic group G the two-sided Fréchet-Stein algebra $D(G, K)$ satisfies* (DIM) *with $d = \dim_L G$.*

The preceding theorem allows to apply the results of the discussion in [loc.cit.], Sect. 8 on an abstract two-sided Fréchet-Stein algebra A satisfying (DIM) directly to $D(G, K)$. Thus, the codimension theory for arbitrary locally $\mathbb{Q}_p$-analytic groups generalizes to locally L-analytic groups as follows.

Lemma 6.2 *Let G be a compact locally L-analytic group, let $D(G, K)$ be endowed with the regular Fréchet-Stein structure and let $M \in \mathcal{C}_G$. Then*

$$j_{D(G,K)}(M) = \min_r j_{D_r(G,K)}\,(D_r(G, K) \otimes_{D(G,K)} M).$$

If $N \subseteq M$ is a coadmissible submodule one has

$$j_{D(G,K)}(M) = \min\ (j_{D(G,K)}(N),\ j_{D(G,K)}(M/N)).$$

Proof: This follows directly from the discussion in [ST5], Sect. 8 on an abstract two-sided Fréchet-Stein algebra satisfying (DIM). □

Lemma 6.3 *Keep the assumptions and suppose additionally that $H \subseteq G$ is an open subgroup. It follows that $j_{D(G,K)}(M) = j_{D(H,K)}(M)$.*

Proof: By Cor. 2.8 we find an open normal subgroup H_1 of G contained in H such that the restriction of H_1 over $\mathbb{Q}_p$ is uniform and satisfies condition (L). We put on $D(H, K)$ and $D(G, K)$ the regular Fréchet-Stein structure induced from $D(H_1, K)$. By [ST5], Lem. 3.8 the $D(G, K)$-module M is then coadmissible over any of these three Fréchet-Stein algebras. Denoting the defining Banach algebras as usual with the subscript $_r$ we have, by the above lemma,

$$j_{D(G,K)}(M) = \min_r j_{D_r(G,K)}\,(D_r(G, K) \otimes_{D(G,K)} M)$$

and similarly for H and H_1. Hence, the claim follows from [ST5], Lem. 8.8 (i) applied to the ring extensions $D_r(H_1,K) \subseteq D_r(G,K)$ and $D_r(H_1,K) \subseteq D_r(H,K)$. $\square$

Now consider an *arbitrary* locally L-analytic group G of dimension d and the abelian category $\mathcal{C}_G$ of coadmissible left $D(G,K)$-modules. By [ST5], Sect. 6 this is the full subcategory of left $D(G,K)$-modules M such that $M \in \mathcal{C}_H$ for some (alternatively every) compact open subgroup $H \subseteq G$. For two such groups H, H' and $M \in \mathcal{C}_G$ we have $j_{D(H,K)}(M) = j_{D(H',K)}(M)$ by the preceding lemma and thus

$$\mathrm{codim}(M) := j_{D(H,K)}(M)$$

is well-defined. If M satisfies $\mathrm{codim}(M) \geq d$ we call M *zero-dimensional*. Furthermore, the property (*) above together with the last lemma yields that the dimension filtration of M as $D(H,K)$-module

$$M = \Delta^0(M) \supseteq \Delta^1(M) \supseteq \dots \supseteq \Delta^{d+1}(M) = 0$$

does not depend on the choice of the compact open subgroup $H \subseteq G$. It is therefore called the *dimension filtration* of the coadmissible $D(G,K)$-module M.

Proposition 6.4 *The dimension filtration of $M \in \mathcal{C}_G$ satisfies:*

i. *Each $\Delta^l(M)$ is a coadmissible $D(G,K)$-submodule of M,*

ii. *a coadmissible $D(G,K)$-submodule $N \subseteq M$ has codimension $\geq l$ if and only if $N \subseteq \Delta^l(M)$,*

iii. $\mathrm{codim}(M) = \sup\{l \geq 0 : \Delta^l(M) = M\}$ *and thus* $\mathrm{codim}(M) \leq d$ *if $M \neq 0$,*

iv. *all nonzero coadmissible $D(G,K)$-submodules of $\Delta^l(M)/\Delta^{l+1}(M)$ have codimension l.*

Proof: Everything is an immediate consequence of [ST5], Prop. 8.7 except the $D(G,K)$-invariance of $\Delta^l(M)$. But this follows from exactly the same argument as in the proof of [loc.cit.], Prop. 8.11. $\square$

As an application we explicitly calculate the codimension in some interesting cases. Over $\mathbb{Q}_p$ all these cases are treated in [ST5], Sect. 8. In proving their analogues over L we use the same ideas but since the arising graded rings are more complicated some extra work is needed. We start with a technical criterion.

Lemma 6.5 *Let H be a compact locally L-analytic group and $J \subseteq D(H,K)$ a left ideal such that $D(H,K)/J$ is coadmissible. If*

$$j_{D_r(H,K)}(D_r(H,K)/D_r(H,K)J) \geq d$$

for all $r \in S(\delta)$ then $D(H,K)/J$ is zero-dimensional.

Proof: By [ST5], Cor. 3.4. the $D(H,K)$-module J is coadmissible, too. The corresponding coherent sheaf is given by the modules

$$D_r(H,K) \otimes_{D(H,K)} J = D_r(H,K)J$$

according to [loc.cit.], Cor. 3.1 and by flatness of $D_r(H,K)$ over $D(H,K)$ ([loc.cit.], Remark 3.2). Using the flatness again we obtain

$$D_r(H,K)/D_r(H,K)J \simeq D_r(H,K) \otimes_{D(H,K)} (D(H,K)/J)$$

as left $D_r(H,K)$-modules. According to Lem. 6.2 the result follows from our assumptions. □

Proposition 6.6 *Let H be a d-dimensional locally L-analytic group such that H_0 is uniform and satisfies condition* (L) *with corresponding L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$, the Lie algebra of H. Let $U(\mathfrak{g}_L)$ be the universal enveloping algebra. Let $\lambda_1, ..., \lambda_d$ be elements of $U(\mathfrak{g}_L)$ such that $\lambda_j = P_j(\mathfrak{x}_j)$ where P_j is a nonzero polynomial in $L[X]$. Let J be the left ideal of $D(H,K)$ generated by the λ_j. Then the coadmissible module $D(H,K)/J$ is zero-dimensional.*

Proof: Since J is finitely generated the $D(H,K)$-modules J and $D(H,K)/J$ are coadmissible by [ST5], Cor. 3.4. It thus suffices, by the preceding lemma, to fix $r \in S(\delta)$ and prove

$$j_{D_r(H,K)}(D_r(H,K)/D_r(H,K)J) \geq d.$$

By Lem. 5.8 we have with $R_1 := D_r(H,K)$, $R_0 := D_\delta(H^{(m)},K)$ that R_1 is free as R_0-bimodule on the finite basis $\mathcal{R}$. Here, $m \in \mathbb{N}_0$ is appropriately chosen, $\mathcal{R}$ is a finite system of representatives for $H^{(m)} \backslash H$ containing 1 and R_0 is a complete filtered ring where the associated graded ring is a polynomial ring over $gr^{\cdot} K$ in d variables. The filtration is as usual induced by the quotient norm of $||.||_\delta^{(m)}$ via the map $D_\delta(H_0^{(m)},K) \to D_\delta(H^{(m)},K)$. We abbreviate this quotient norm by $||.||_{\bar{\delta}}$. Since $H^{(m)}$ is open in H we may identify the Lie algebras and then consider the left ideal J_0 generated by $\lambda_1, ..., \lambda_d$ inside R_0. Abbreviate $J_1 := D_r(H,K)J$ and consider the surjective homomorphism $R_1 \otimes_{R_0} J_0 \longrightarrow R_1 J_0 = J_1$ of left R_1-modules. It is bijective by flatness of R_1 over R_0. Using the flatness again we obtain

$$R_1/J_1 = R_1 \otimes_{R_0} (R_0/J_0)$$

as left R_1-modules. By Lem. 5.4 and faithful flatness of R_1 over R_0 we obtain

$$j_{R_1}(R_1/J_1) = j_{R_0}(R_0/J_0).$$

We are thus reduced to prove that the right-hand side is $\geq d$. Since the filtration induced by the norm $||.||_{\bar{\delta}}$ on R_0 is quasi-integral we have

$$gr^{\cdot}_{\delta}\,(R_0/J_0) \simeq gr^{\cdot}_{\delta}\,R_0/gr^{\cdot}_{\delta}\,J_0$$

as $gr^{\cdot}_{\delta}\,R_0$-modules where the left-hand side carries the quotient filtration. Furthermore, R_0 is a Zariski ring ([LVO], Thm. II.2.2.1) and so the induced filtrations on J_0 and R_0/J_0 are good ([LVO], Thm. II.2.1.1). Using that $gr^{\cdot}_{\delta}\,R_0$ is Auslander regular we may deduce

$$j_{R_0}(R_0/J_0) = j_{gr^{\cdot}_{\delta}\,R_0}(gr^{\cdot}_{\delta}\,(R_0/J_0)) = j_{gr^{\cdot}_{\delta}\,R_0}(gr^{\cdot}_{\delta}\,R_0/gr^{\cdot}_{\delta}\,J_0)$$

([LVO], Thm. III.2.5.2). Now $R := gr^{\cdot}_{\delta}\,R_0$ is a commutative (Auslander) regular and catenary noetherian domain of Krull dimension $d+1$. By [BH], Cor. 3.5.11 we have for any finitely generated R-module N the formula

$$j_R(N) = d + 1 - \text{ Krulldim}(R/\text{ann}(N))$$

where $\text{ann}(N)$ denotes the annihilator of N in R. Taking $N := gr^{\cdot}_{\delta}\,R_0/gr^{\cdot}_{\delta}\,J_0$ we are reduced to prove that the factor ring $gr^{\cdot}_{\delta}\,R_0/gr^{\cdot}_{\delta}\,J_0$ has Krull dimension ≤ 1.

Now recall from Prop. 3.20 that $R_0 = D_\delta(H^{(m)}, K)$ as a K-Banach space is given by all series

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with coefficients $d_\alpha \in K$ such that $|d_\alpha|\delta^{\kappa|\alpha|} \rightarrow 0$ for $|\alpha| \rightarrow \infty$. Here, $\mathbf{b}^\alpha := b_1^{\alpha_1} \cdots b_d^{\alpha_d}$, b_j is the image of $b_{1j} := h_{1j} - 1$ under the quotient map $D_\delta(H_0^{(m)}, K) \rightarrow D_\delta(H^{(m)}, K)$ and $h_{11}, ..., h_{nd}$ is an ordered basis of the uniform group $H_0^{(m)}$. Moreover, the norm $||.||_{\bar{\delta}}$ on R_0 is computed on a series λ as above via

$$||\lambda||_{\bar{\delta}} = \sup_\alpha ||d_\alpha \mathbf{b}^\alpha||_{\bar{\delta}} = \sup_\alpha |d_\alpha|\delta^{\kappa\,|\alpha|}. \tag{38}$$

As graded ring we have

$$gr^{\cdot}_{\delta}\,R_0 = (gr^{\cdot} K)[\sigma(b_1), ..., \sigma(b_d)]$$

where σ denotes the principal symbol map. Now identifying the Lie algebras of $H^{(m)}$ and $H_0^{(m)}$ over $\mathbb{Q}_p$ as usual we have $\mathfrak{x}_j = \log(1 + b_{1j})$ in $D_\delta(H_0^{(m)}, K)$ by (the proof of) Prop. 3.6 and hence $\mathfrak{x}_j = \log(1 + b_j)$ in $D_\delta(H^{(m)}, K) = R_0$. Hence, in $gr^\cdot_\delta R_0$ we obtain $\sigma(\mathfrak{x}_j) = \sigma(b_j)$ by (38) and $\delta^\kappa < p^{-\frac{1}{p-1}}$ and therefore

$$\sigma(\lambda_j) = \sigma(P_j(\mathfrak{x}_j)) = P_j'(\sigma(\mathfrak{x}_j)) = P_j'(\sigma(b_j))$$

with some nonzero polynomial $P_j' \in (gr^\cdot L)[X]$. Since $\sigma(\lambda_j) \in gr^\cdot_\delta J_0$ we have a surjection

$$(gr^\cdot K)[\sigma(b_1)]/(\sigma(\lambda_1)) \otimes_{gr^\cdot K} ... \otimes_{gr^\cdot K} (gr^\cdot K)[\sigma(b_d)]/(\sigma(\lambda_d)) \to gr^\cdot_\delta R_0/gr^\cdot_\delta J_0.$$

Each $(gr^\cdot K)[\sigma(b_j)]/(\sigma(\lambda_j))$ is finitely generated as a module over the ring $gr^\cdot K$ and hence, so is $gr^\cdot_\delta R_0/gr^\cdot_\delta J_0$. But $gr^\cdot K$ has Krull dimension 1 being Laurent polynomials over k, the residue field of K (cf. 3.3). By [B-CA], Prop. V.2.1.1 and Cor. V.2.1.1 the ring $gr^\cdot_\delta R_0/gr^\cdot_\delta J_0$ has then Krull dimension ≤ 1. $\square$

Theorem 6.7 *Let G be a d-dimensional locally L-analytic group and $M \in \mathcal{C}_G$. If the action of the universal enveloping algebra $U(\mathfrak{g}_L)$ on M is locally finite, i.e., if $U(\mathfrak{g}_L)x$, for any $x \in M$, is a finite dimensional L-vectorspace then M is zero-dimensional.*

Proof: Fix a compact open subgroup $H \subseteq G$ such that H_0 is uniform satisfying (L). In order to prove $M \subseteq \Delta^d(M)$ it suffices to check, for fixed $x \in M$, that the cyclic $D(H,K)$-module $D(H,K)x$ lies in $\Delta^d(M)$. By [ST5], Cor. 3.4 iv. this amounts to proving that $D(H,K)x \in \mathcal{C}_H$ is zero-dimensional itself.

Let $\mathfrak{x}_1, ...\mathfrak{x}_d$ be a basis of $\mathfrak{g}_L$ realising condition (L) for H (we identify $\mathfrak{g}_L$ with the Lie algebra of H as usual). Write

$$D(H,K)x = D(H,K)/J_1$$

with some left ideal J_1 of $D(H,K)$ and let J_0 be the kernel ideal of $U(\mathfrak{g}_L) \to U(\mathfrak{g}_L)x$. Then $J_0 \subseteq J_1$ and the left ideal J_0 of $U(\mathfrak{g}_L)$ has, by assumption, finite codimension in $U(\mathfrak{g}_L)$. Hence, J_0 must contain, for every $j = 1, ..., d$, an element $\lambda_j := P_j(\mathfrak{x}_j)$ where $P_j \in L[X]$ is a nonzero polynomial. The left ideal $J \subseteq J_1$ generated by these elements λ_j in $D(H,K)$ satisfies the assumptions of the preceding proposition. Thus $D(H,K)/J$ is zero-dimensional and hence, so is its quotient $D(H,K)/J_1$ according to Lem. 6.2. $\square$

Let as before G be a locally L-analytic group. Recall that a *smooth* G-representation V over K is a K-vectorspace V with a linear G-action such that the stabilizer of each vector $v \in V$ is open in G.

A smooth G-representation is called *admissible-smooth* if, for any compact open subgroup $H \subseteq G$, the vector subspace V^H of H-invariant vectors in V is finite dimensional. An admissible-smooth G-representation V equipped with the finest locally convex topology is admissible in the sense that its strong dual lies in $\mathcal{C}_G$ ([ST5], Thm. 6.6).

Corollary 6.8 *If V is an admissible-smooth G-representation then the corresponding coadmissible $D(G, K)$-module is zero-dimensional.*

Proof: By [ST5], Thm. 6.6 the derived Lie algebra action on V is trivial. The corresponding coadmissible module M is the strong dual V'_b with $D(G, K)$-action induced by the contragredient G-action ([loc.cit.] Thm. 6.3). Thus $U(\mathfrak{g}_L)x = Lx$ for all $x \in M$ and the preceding theorem applies. □

Theorem 6.9 *Assume G is a compact. Let $M \in \mathcal{C}_G$. If the K-vectorspace*

$$M_r := D_r(G, K) \otimes_{D(G,K)} M$$

is finite-dimensional for all $r \in S(\delta)$ then M is zero-dimensional.

Proof: Fix an open normal subgroup H of G such that H_0 is uniform satisfying condition (L) with respect to a basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$, the Lie algebra of H. With the same argument as above it suffices to check that the cyclic $D(H, K)$-module $D(H, K)x \in \mathcal{C}_H$ is zero-dimensional for every $x \in M$. Furthermore, writing $D(H, K)x = D(H, K)/J$ with some left ideal J it suffices, by the preceding lemma, to prove

$$j_{D_r(H,K)}\left(D_r(H, K)/D_r(H, K)J\right) \geq d$$

for all $r \in S(\delta)$.

Since $D_r(H, K)$ is flat over $D(H, K)$ according to [ST5], Remark 3.2 we have an inclusion of $D_r(H, K)$-modules

$$D_r(H, K)/D_r(H, K)J = D_r(H, K) \otimes_{D(H,K)} D(H, K)x \longrightarrow M_r$$

and thus the left-hand side is a finite-dimensional K-vectorspace. Letting $R_1 := D_r(H, K),\ J_1 := D_r(H, K)J,\ J_0 := (K \otimes_L U(\mathfrak{g}_L)) \cap J_1$ we have the inclusion of K-vectorspaces

$$K \otimes_L U(\mathfrak{g}_L)\ /J_0 \longrightarrow R_1/J_1$$

and hence, J_0 has finite codimension in $K \otimes_L U(\mathfrak{g}_L)$. From here on, the proof is exactly (modulo the scalar extension from L to K which is harmless) the same as in the preceding theorem. $\square$

7 Properties of the duality functor

In this section we briefly indicate how parts of the duality theory developed by Schneider/Teitelbaum in [ST6] extend from the base field $\mathbb{Q}_p$ to a finite extension $L/\mathbb{Q}_p$. This theory addresses the problem of finding an involutive "duality" auto-functor on the category of admissible representations which is compatible with the classical smooth contragredient on the subcategory of all admissible-smooth representations.

We remark straightaway that most of the theory contained in [loc.cit.] is already available over the base field L. However, in proving the involutivity the authors have to use the regularity properties of the Fréchet-Stein algebra $D(G,K)$ (as exhibited in [ST5], Thm. 8.9) and thus have to restrict to the base field $\mathbb{Q}_p$. Furthermore, they also compute the explicit shape of the functor on the quotient subcategories of $\mathcal{C}_G$ corresponding to the grade filtration and at this point, have to restrict to $\mathbb{Q}_p$ as well.

Now the proofs of these results carry over to the general base field L in an almost unchanged manner using the fact that the key result Thm. 8.9 of [ST5] generalizes to Thm. 6.1 above. We will recall as much as necessary from [ST6] in order to give a precise statement.

Let G be as usual a locally L-analytic group of dimension d and $\mathfrak{g}_L$ its Lie algebra. Let $D(G,K)$ resp. $D^\infty(G,K)$ be the algebras of locally analytic resp. locally constant K-valued distributions on G. Write $\mathcal{M}_G$ resp. $\mathcal{M}_G^\infty$ for the categories of left modules over $D(G,K)$ resp. $D^\infty(G,K)$. Let $\mathcal{D}_K(G) \in \mathcal{M}_G$ be the locally analytic dualizing module as constructed in [ST6], Sect. 2. Let $\mathrm{Rep}_K^a(G)$ resp. $\mathrm{Rep}_K^{\infty,a}(G)$ be the categories of admissible resp. admissible-smooth G-representations and $\mathcal{C}_G$ resp. $\mathcal{C}_G^\infty$ the associated categories of coadmissible modules.

For any abelian category $\mathcal{A}$ we let $D^b(\mathcal{A})$ be the bounded derived category (meaning the derived category of all complexes in $\mathcal{A}$ with only finitely many nonzero cohomology objects). Whenever $\mathcal{A}_0 \subseteq \mathcal{A}$ is a full abelian subcategory closed under extensions we have the triangulated subcategory $D^b_{\mathcal{A}_0}(\mathcal{A})$ of $D^b(\mathcal{A})$ consisting of all those complexes whose cohomology objects lie in $\mathcal{A}_0$. The "bounded below" versions of these categories are as usual denoted by replacing the superscript "b" by "$+$". Let us remark that the full abelian subcategory $\mathcal{C}_G \subseteq \mathcal{M}_G$ is closed under extensions (by [ST5], Remark 3.2 and Cor. 3.1) so that the triangulated subcategory $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ is at our disposal.

Denote by

$$.^* : \ \mathrm{Rep}_{\mathrm{K}}^{\infty,\mathrm{a}}(G) \to \mathcal{M}_G^\infty \qquad \text{resp.}$$

$$.^\sim : \ \mathrm{Rep}_{\mathrm{K}}^{\infty,\mathrm{a}}(G) \to \mathrm{Rep}_{\mathrm{K}}^{\infty,\mathrm{a}}(G)$$

the functors "passing to the full linear dual" resp. "passing to the smooth dual". Since these two functors as well as the natural functor $\mathcal{M}_G^\infty \to \mathcal{M}_G$ are all exact they pass directly to derived categories where we denote them by the same symbols.

Denote by

$$\delta_G : G \to \mathbb{Q}^\times \subseteq K^\times$$

the locally constant modulus character and let δ_G denote also the one-dimensional K-vectorspace K viewed as $D^\infty(G,K)$-bimodule where G acts trivially from the left and through the character δ_G from the right. We also write δ_G for its image in $\mathcal{M}_G$. Put

$$\Delta_G := \bigwedge^d \mathfrak{g}_L \otimes_L K,$$

considered as a one-dimensional locally analytic G-representation. This diagonal action extends uniquely to a separately continuous left $D(G,K)$-module structure (via the comultiplication in $D(G,K)$, cf. [loc.cit.], Sect. 3 App.). Finally, put

$$\mathfrak{d}_G := \Delta_G \otimes_K \delta_G^*$$

viewed as a $D(G,K)$-bimodule with $D(G,K)$ acting trivially from the right and through the product character $\Delta_G \cdot \delta_G$ from the left. Then

$$\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G) \otimes_K \mathfrak{d}_G)$$

is an auto-functor on $\mathcal{M}_G$ (where the left $D(G,K)$-module structure on $\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G) \otimes_K \mathfrak{d}_G)$ comes from the right multiplication on the target together with the usual anti-involution on $D(G,K)$).

Proposition 7.1 *Restricting the functor*

$$R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) : D^+(\mathcal{M}_G) \longrightarrow D^+(\mathcal{M}_G)$$

to the subcategory $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ induces a commutative diagramm of functors

$$\begin{array}{ccc} D^b(\mathrm{Rep}_{\mathrm{K}}^{\infty,\mathrm{a}}(G)) & \xrightarrow{.^*} & D^b_{\mathcal{C}_G}(\mathcal{M}_G) \\ {\scriptstyle .^\sim[-d]}\downarrow & & \downarrow{\scriptstyle R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G)\otimes_K \mathfrak{d}_G)} \\ D^b(\mathrm{Rep}_{\mathrm{K}}^{\infty,\mathrm{a}}(G)) & \xrightarrow{.^*} & D^b_{\mathcal{C}_G}(\mathcal{M}_G). \end{array}$$

Here, [.] *denotes, as usual, a degree shift.*

Proof: For the base field $\mathbb{Q}_p$ this is precisely the statement of [ST6], Cor. 4.2. For the base field L it follows like this. As a consequence of [ST6], Cor. 3.7 one has the commutativity of the square with the right lower corner replaced by $D^+(\mathcal{M}_G)$. Now the argument in the proof of [ST6], Cor. 4.2 reduces us to show that for any complex $X^\cdot$ in $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ the $D(G,K)$-modules $\mathrm{Ext}^*_{D(G,K)}(X^\cdot, \mathcal{D}_K(G))$ are in $\mathcal{C}_G$ and vanish in all but finitely many degrees. But this is precisely the statement of [loc.cit.], Prop. 4.1.(i) whose proof carries over to the base field L using that [ST5], Thm. 8.9 admits the generalization Thm. 6.1. $\square$

Proposition 7.2 *The functor* $R\mathrm{Hom}_{D(G,K)}(., \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G)$ *on* $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ *is an anti-involution.*

Proof: For the base field $\mathbb{Q}_p$ this is precisely the statement [ST6], Cor. 4.4 whose proof is based on [loc.cit.], Prop. 4.3. But using the generalization Thm. 6.1 of [ST5], Thm. 8.9 the proof of this latter proposition carries over to the base field L whence the analogue of Cor. 4.4 over L follows. $\square$

We conclude by pointing out how the duality functor is computed on certain abelian subquotient categories of $\mathcal{C}_G$. According to the results of the last section, for any $l \in \mathbb{N}_0$, we have the full subcategory $\mathcal{C}^l_G$ in $\mathcal{C}_G$ of all coadmissible modules of codimension $\geq l$ available. Note that $\mathcal{C}^\infty_G \subseteq \mathcal{C}^d_G$ by Cor. 6.8. By Lem. 6.2 each $\mathcal{C}^l_G \subseteq \mathcal{C}_G$ is closed under subobjects, quotients and extensions, i.e.

$$\mathcal{C}_G = \mathcal{C}^0_G \supseteq \mathcal{C}^1_G \supseteq ... \supseteq \mathcal{C}^{d+1}_G = 0$$

is a filtration of the abelian category $\mathcal{C}_G$ by Serre subcategories. We may thus form the abelian quotient categories $\mathcal{C}^l_G/\mathcal{C}^{l+1}_G$. By [ST6], Lem. 5.1 (whose proof carries over to the base field L using Thm. 6.1)

$$\mathrm{Ext}^l_{D(G,K)}(., \mathcal{D}_K(G)) : \mathcal{C}^l_G \longrightarrow \mathcal{C}^l_G/\mathcal{C}^{l+1}_G$$

is a well-defined and exact functor being zero on $\mathcal{C}^{l+1}_G$.

Proposition 7.3 *The duality functor descends to the auto-functor*

$$\mathrm{Ext}^l_{D(G,K)}(., \mathcal{D}_K(G)) : \mathcal{C}^l_G/\mathcal{C}^{l+1}_G \longrightarrow \mathcal{C}^l_G/\mathcal{C}^{l+1}_G$$

on the subquotient category $\mathcal{C}^l_G/\mathcal{C}^{l+1}_G$*. In particular,* $\mathrm{Ext}^l_{D(G,K)}(., \mathcal{D}_K(G))$ *is an anti-involution on* $\mathcal{C}^l_G/\mathcal{C}^{l+1}_G$*.*

Proof: This is [ST6], Prop. 5.2 for the base field $\mathbb{Q}_p$. Its proof relies on [loc.cit.], Prop. 4.3, Lem. 5.1 and thus carries over to the base field L. $\square$

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