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› Acylindrical Actions on Trees and the Farrell–Jones Conjecture

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Abstract

This thesis is concerned with groups acting acylindrically on simplicial trees and a relative version of the K - and L -theoretic Farrell–Jones Conjecture. We show that for such groups the conjecture holds relative to the family of subgroups consisting of virtually cyclic subgroups and all subconjugates of vertex stabilisers. Moreover, for amalgamated free products acting acylindrically on their Bass-Serre trees we obtain an identification of the associated Waldhausen Nil-groups with a direct sum of cokernels of relative assembly maps for certain virtually cyclic groups. For a regular ring and a strictly acylindrical action, these groups vanish. In particular, all these results hold for amalgamated free products over malnormal subgroups.

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Introduction

For a group G , both the Whitehead group $\text{Wh}(G) = K_1(\mathbb{Z}G)/\langle \pm g \mid g \in G \rangle$ and the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$ of the group ring $\mathbb{Z}G$ are the habitat of important obstructions in geometric topology:

For a closed connected oriented smooth manifold M of dimension at least 5, the s-cobordism theorem due to Barden, Mazur and Stallings (cf. [Ker65]) states that an h -cobordism over M is diffeomorphic to the cylinder $M \times I$ if and only if its Whitehead torsion (a particular element in the Whitehead group of $\pi_1(M)$) is trivial. Moreover, the h -cobordisms over M are (up to diffeomorphism) in one-to-one correspondence with elements in $\text{Wh}(\pi_1(M))$ by assigning to each such cobordism its Whitehead torsion. Wall's finiteness obstruction on the other hand, introduced by Wall in [Wal65], lives in $\tilde{K}_0(\mathbb{Z}G)$ and detects whether or not a given finitely dominated (connected) space with fundamental group G is homotopy equivalent to a finite CW-complex.

This makes computing algebraic K -groups desirable for topologists. Unfortunately, computing these groups tends to be rather involved, and assembly-techniques, aiming to utilise knowledge about the K -theory of group rings RH for suitable subgroups $H \leq G$ in order to compute $K_*(RG)$, have been in the focus of research for a long time.

A general set-up to potentially compute $K_*(RG)$ from smaller 'building blocks' is provided by the Farrell–Jones Conjecture relative to a family $\mathcal{F}$ of subgroups of G : A *family $\mathcal{F}$ of subgroups of G* is a collection of subgroups of G which is closed under conjugation and taking subgroups. Examples are the family $\mathcal{FIN}$ of all finite subgroups or the family $\mathcal{VCC}$ of all virtually cyclic subgroups of G . For each group G and every ring R (with involution) there are G -homology theories $H_*^G(-; \mathbf{K}_R)$ and $H_*^G(-; \mathbf{L}_R^{-\infty})$ by [DL98] with the following key feature: their values on each homogeneous space G/H are the K - and L -theory, respectively, of the group ring RH , i.e.

$$H_*^G(G/H; \mathbf{K}_R) \cong K_*(RH) \quad \text{and} \quad H_*^G(G/H; \mathbf{L}_R^{-\infty}) \cong L_*^{\langle -\infty \rangle}(RH)$$

for all subgroups H of G . The (*K - and L -theoretic*) *Farrell–Jones Conjecture for G relative to a family of subgroups $\mathcal{F}$* predicts that the *assembly maps*

$$\begin{aligned} H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) &\rightarrow H_*^G(\text{pt}; \mathbf{K}_R) \cong K_*(RG) \quad \text{and} \\ H_*^G(E_{\mathcal{F}}G; \mathbf{L}_R^{-\infty}) &\rightarrow H_*^G(\text{pt}; \mathbf{L}_R^{-\infty}) \cong L_*^{\langle -\infty \rangle}(RG), \end{aligned}$$

induced by the projection $E_{\mathcal{F}}G \rightarrow \text{pt}$, are isomorphisms in all degrees. Here, $E_{\mathcal{F}}G$ is the classifying space for the family $\mathcal{F}$. (Any G -CW-complex X with $X^H \simeq \text{pt}$ for $H \in \mathcal{F}$ and $X^H = \emptyset$ for $H \notin \mathcal{F}$ is a model for $E_{\mathcal{F}}G$.)

The Farrell–Jones Conjecture relative to $\mathcal{VCYC}$ is known for a large class of groups including hyperbolic groups [BLR08a, BL12a], $\text{CAT}(0)$ -groups [BL12a, Weg12], virtually solvable groups [Weg15] and $\text{GL}_n(\mathbb{Z})$ [BLRR14]. If the K -theoretic version of the conjecture holds for a torsion-free group G , then both $\text{Wh}(G)$ and $\tilde{K}_0(\mathbb{Z}G)$ vanish. If both the K - and L -theoretic version hold, then the Borel Conjecture for G follows. A detailed exposition on these and further applications can be found in [BLR08b] and [LR05].

In this thesis we investigate the Farrell–Jones Conjecture for groups acting acylindrically on simplicial trees: A discrete group G acts *k-acylindrically* on a simplicial tree T if the pointwise stabiliser of every geodesic segment of length k in T is finite (we do not require these stabilisers to be trivial or to have uniform cardinality). An action $G \curvearrowright T$ is called *acylindrical* if it is *k-acylindrical* for some k . Examples of acylindrical actions can be given by considering an amalgamated free product $G = A *_C B$, where C is an almost malnormal subgroup in A or B (see Section 1.2).

In general, for a group action $G \curvearrowright T$ on a simplicial tree, denote by $\mathcal{F}_T$ the family of subgroups of G which are subconjugate to vertex stabilisers, and by $\mathcal{F}_{\partial}$ the family of subgroups of G that fix a pair of boundary points of T pointwise (cf. Chapter 2). The first main theorem of this thesis is the following.

Theorem 3.1. *Let G be a group acting acylindrically on a simplicial tree T and let the family $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_{\partial}$ be as above. Let $\mathcal{F}_2$ be the family of subgroups of G that contain a group in $\mathcal{F}$ as a subgroup of index ≤ 2 . Then*

- a) *G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}$;*
- b) *G satisfies the L -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}_2$.*

In the special cases of 0- or 1-acylindrical actions this theorem was already known: If G acts 0-acylindrically on a tree T (and G is finitely generated), then G is hyperbolic and the Farrell–Jones Conjecture relative to $\mathcal{VCYC} \subset \mathcal{F}_T \cup \mathcal{F}_{\partial}$ is known. A recent result of Bartels [Bar16] for relatively hyperbolic groups covers, in particular, the 1-acylindrical case. We will see in Section 1.2 that it is easy to construct actions that are acylindrical but not 1-acylindrical.

If the vertex stabilisers of the action $G \curvearrowright T$ are known to satisfy the Farrell–Jones Conjecture relative to $\mathcal{VCYC}$, then inheritance properties of the conjecture allow us to obtain the following corollary and its L -theoretic analogue, which is formulated in Corollary 3.5.

Corollary 3.4. *Let G be a group acting acylindrically on a simplicial tree T . If all vertex stabilisers of $G \curvearrowright T$ satisfy the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$, then G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$.*

The Farrell–Jones Conjecture (with coefficients in additive categories) relative to $\mathcal{VCYC}$ has various useful inheritance properties. For example, if two groups A and B satisfy the conjecture, then so do $A \times B$ and $A * B$. If A and B are abelian, then an amalgamated free product $G = A *_C B$ satisfies the Farrell–Jones Conjecture relative to $\mathcal{VCYC}$ by [GMR15], but for a general amalgamated free product this inheritance property is not known. In this context the last corollary can also be interpreted as follows: Any amalgamated free product $G = A *_C B$ with C being almost malnormal in either A or B acts acylindrically on its Bass-Serre tree. Thus, the above corollary implies that the class of groups satisfying the Farrell–Jones Conjecture relative to $\mathcal{VCYC}$ is closed under taking amalgamated free products over an almost malnormal subgroup.

Another angle on amalgamated free products and the computation of algebraic K -theory is the following: For an amalgamated free product $G = A *_C B$ one can ask whether there is a Mayer-Vietoris type exact sequence

$$\dots \rightarrow K_n(RC) \rightarrow K_n(RA) \oplus K_n(RB) \rightarrow K_n(RG) \rightarrow K_{n-1}(RC) \rightarrow \dots$$

In general this is not the case, and Waldhausen famously introduced an exact category $\mathfrak{Nil}(RC; R[A - C], R[B - C])$ encoding to what extent exactness of the above sequence fails in [Wal78a, Wal78b]. In particular, *Waldhausen Nil-groups* $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ associated to the amalgamated free product $G = A *_C B$ are direct summands in the homotopy groups of a certain non-connective spectrum version of $\mathfrak{Nil}(RC; R[A - C], R[B - C])$, and the sequence

$$\dots \rightarrow K_n(RC) \rightarrow K_n(RA) \oplus K_n(RB) \rightarrow K_n(RG)/\widetilde{\text{Nil}}_{n-1} \rightarrow K_{n-1}(RC) \rightarrow \dots$$

becomes exact. Naturally, one is interested in identifying $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ with something that is known to vanish often.

For an amalgamated free product $G = A *_C B$ that acts acylindrically on the associated Bass-Serre tree *and* is already known to satisfy the Farrell–Jones Conjecture relative to $\mathcal{VCYC}$, Lafont and Ortiz obtained an identification in [LO09]. Building on Theorem 3.1, we are able to lift the second assumption. Namely, we obtain the following generalisation of their theorem.

Theorem 3.9. *Let $G = A *_C B$ act acylindrically on the associated Bass-Serre tree T . Let $\mathcal{L}$ be a set of representatives for the orbits of the action $G \curvearrowright \partial T \times \partial T \setminus \text{diag}$. Then for any ring R there are isomorphisms*

$$\widetilde{\text{Nil}}_{n-1}(RC; R[A - C], R[B - C]) \cong \bigoplus_{L \in \mathcal{L}} \text{coker}(H_n^{G_L}(E_{\mathcal{FIN}} G_L; \mathbf{K}_R) \rightarrow H_n^{G_L}(\text{pt}; \mathbf{K}_R))$$

for $n \in \mathbb{Z}$.

More can be said about the right hand side of the above isomorphisms: The groups G_L appearing in the theorem are in fact virtually cyclic of type I. For any such finite-by- $\mathbb{Z}$ group $V = H \rtimes_{\alpha} \mathbb{Z}$ it is known (see [DQR11, Lemma 3.1]) that the cokernel of the relative assembly map $H_n^V(E_{\mathcal{FIN}}V; \mathbf{K}_R) \rightarrow H_n^V(E_{\text{VCYC}}V; \mathbf{K}_R)$ is isomorphic to the direct sum of two Farrell Nil-groups associated to RH and α . These type of Nil-groups were introduced by Farrell in his PhD-thesis [Far71] (for a modern definition encompassing all degrees see [Gru08]). In the case present, where H is finite, these Farrell Nil-groups are known to be either trivial or infinitely generated: in lower degrees, this was shown independently by Grunewald [Gru07] and Ramos [Ram07]. Very recently, Lafont, Prassidis and Wang extended this result to all degrees [LPW16] and obtained a structure result for these groups (provided they have finite exponent). For a regular ring R , the Farrell Nil-groups constituting the cokernel of the relative assembly map for virtually cyclic groups of type I are also known to be torsion: for $R = \mathbb{Z}$ this is due to Kuku and Tang [KT03]. For an arbitrary regular ring R this follows from [LS15, Theorem 0.3].

Lastly, exploiting what is known about the relative assembly map, we collect the following vanishing results for Waldhausen Nil-groups.

Corollary 3.10. *Let $G = A *_C B$ act acylindrically on its Bass-Serre tree T and let R be a regular ring. Then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ vanishes rationally. Furthermore,*

- a) *if the action $G \curvearrowright T$ is strictly acylindrical (e.g. when C is malnormal in A or B), then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C]) = 0$;*
- b) *if $\mathbb{Q} \subset R$, then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C]) = 0$.*

Next we give the structure of this thesis and, while doing so, elaborate in detail on the ideas used to prove Theorem 3.1.

Chapter 1 encompasses the necessary preliminaries. In order to prove the Farrell–Jones Conjecture relative to a family $\mathcal{F}$ it suffices to show that G admits a finitely $\mathcal{F}$ -amenable action on a compact contractible controlled finitely-dominated metric space. This fact and these notions are explained in Section 1.1. In Section 1.2 we introduce acylindrical actions on simplicial trees and collect a wide variety of examples thereof, whereas Section 1.3 provides background on the observers’ topology and related terms. In particular, the latter section includes a proof of the fact that the Bowditch compactification of a countable tree is a compact contractible controlled finitely-dominated metric space. This reduces the proof of Theorem 3.1 to showing that for an acylindrical action $G \curvearrowright T$, the induced action of G on the Bowditch compactification of T is finitely $\mathcal{F}_T \cup \mathcal{F}_{\partial}$ -amenable.

In Chapter 2, we verify, building on methods of Bartels for relatively hyperbolic groups [Bar16], that the action of G on the Bowditch boundary $\Delta(T)$ is finitely $\mathcal{F}_T \cup \mathcal{F}_{\partial}$ -amenable (see Theorem 2.2). This is the major step towards Theorem 3.1. Showing $\mathcal{F}$ -amenability of the action $G \curvearrowright \Delta(T)$ amounts to providing intricate ‘wide’ covers for $G \times \Delta(T)$. A group G acting 1-acylindrically on a tree is relatively hyperbolic, so in this case the desired covers are already given by Bartels’ methods:

In [Bar16], first a notion of Θ -small angles ‘between edges’ is introduced. Its crucial feature is that it functions like a proper G -invariant metric on the set of edges $E(T)$ incident to a given vertex. Then, a geodesic is called Θ -small if the angle between any two incident edges lying on this geodesic is Θ -small. In particular, any Θ -small geodesic ending in a vertex can only be extended in finitely many directions without losing the property of being Θ -small. Bartels uses the notion of Θ -small geodesics to divide $G \times \Delta(T)$ in two subsets. For trees, the first one, $G \times_{\Theta} \partial T$, consists of all pairs $(g, \xi) \in G \times \partial T$ such that, starting from a fixed vertex in T of finite valence, the point $g^{-1}\xi$ can be reached with a Θ -small geodesic ray. So this set is of the form $G \cdot (\{1\} \times \partial T')$ for a locally finite subtree $T' \subset T$. Furthermore, by definition of a relatively hyperbolic group, G acts cocompactly on $E(T)$ and each edge stabiliser is finite. In the proof of Bartels, these three facts play an essential role in showing that $G \times_{\Theta} \partial T$ admits suitable covers. The second set, the rest, is covered by specially tailored sets that are constructed ad hoc, using the same notion of Θ -small angles.

In our case of groups acting k -acylindrically on trees, edge stabilisers are in general not finite. The entities that have finite stabilisers instead are the geodesic segments of length k in T . In trying to define a notion of Θ -small geodesics by measuring ‘angles’ between ‘incident segments of length k ’ on a geodesic, several difficulties arise. First, from a naïve geometric point of view, it is not clear when two segments of length k should be called incident. The second problem appears when determining which points $g^{-1}\xi \in \partial T$ can be reached from a given fixed vertex v_0 . Since on a geodesic of length $< k$ there is no segment of length k , and thus no ‘angle’, any such geodesic is automatically Θ -small. Hence, one can start from v_0 in infinitely many directions along a Θ -small geodesic, and the set $G \times_{\Theta} \partial T$ would be of the form $G \cdot (\{1\} \times \partial T')$ where T' is a tree that is *not* locally finite. Moreover, the action of G on the set $E^k(T)$ of geodesic segments of length k is not cocompact anymore. We resolve these difficulties as follows.

In Section 2.1, a proper G -invariant metric on $E^k(T)$ is defined. This metric is constructed to have the additional property that the action of G on $E^k(T)$ is ‘cocompact along Θ -small geodesics’, and this property is sufficient for our purposes. Using this metric we can measure the distance between any two segments of length k . Our notion of Θ -small geodesics is defined in Section 2.2, where ‘incident segments of length k ’ on a geodesic are segments on this geodesic whose midpoints have distance 1 in T . For the set $G \times_{\Theta} \partial T$, instead of starting to measure from a single vertex, we start to measure from one of two fixed geodesic segments of length k . These segments are ensured to be connected by a Θ -small geodesic. This way $G \times_{\Theta} \partial T$ can again be written as $G \cdot (\{1\} \times \partial T')$ for a locally finite subtree T' of T , and wide covers for $G \times_{\Theta} \partial T$ are constructed in Sections 2.5–2.7 utilising the long thin covers for coarse flow spaces provided by [Bar16, Theorem 1.1].

Covers for the remaining set $G \times \Delta(T) \setminus G \times_{\Theta} \partial T$ are defined ad hoc in Section 2.3. In general, a single such cover will not be wide, but in Section 2.4 we show that one can combine $k + 2$ of these covers to obtain a wide cover for $G \times \Delta(T) \setminus G \times_{\Theta} \partial T$. Since ‘incident segments of length k ’ overlap, this is more involved than in the case $k = 1$, and we elaborate on the difficulties that arise from overlapping segments at the beginning of Section 2.4.

Chapter 3 is dedicated to the main results of this thesis. The results about the Farrell–Jones Conjecture for acylindrical group actions on trees are provided in Section 3.1 and based on Theorem 3.1 the results on Waldhausen Nil-groups are deduced in Section 3.2.

Appendix A contains a tangent on the Farrell–Jones Conjecture and directed colimits, which is used to reduce the proof of Theorem 3.1 to the case of countable groups acting acylindrically on countable trees.

Conventions:

All groups throughout this thesis are discrete. Any countable group occurring in this thesis is assumed to be equipped with an implicitly chosen proper (left-)invariant metric—indeed, a discrete group is countable if and only if it admits a proper (left-)invariant metric (see for instance [Str74]).

Given a metric space (X, d) , for $r > 0$ and $x \in X$, we denote—if not explicitly stated otherwise—by $B_r^d(x) = B_r(x)$ the *closed* ball of radius r around x .

By a generalised metric on X we mean a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ which is symmetric, satisfies the triangle inequality and $d(x, y) = 0$ if and only if $x = y$.

Finally, we stress that it is hard to draw an unbounded geodesic ray or a tree which is not locally finite. So all figures appearing in this thesis have to be regarded as ‘incomplete illustrations’ that hopefully still capture the essence of the notion or proof at hand.

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Chapter 1

Preliminaries

This chapter provides background on the Farrell–Jones Conjecture (Section 1.1), on acylindrical actions (Section 1.2) and on the observers’ topology (Section 1.3). The first two sections focus on collecting information from the literature needed at central points later on in this thesis. Opposed to this, the third section contains detailed proofs—some of them added as a kind of warm-up, others for the sake of completeness.

1.1 The Farrell–Jones Conjecture

The Farrell–Jones Conjecture predicts an isomorphism between the K -(resp. L -)groups of a group ring RG and the value of a certain G -homology theory on a certain classifying space. It goes back to Farrell and Jones [FJ93, 1.6, p. 257]. Since then its formulation has been further conceptualised and broadly generalised, and its present form (with coefficients in additive categories) was introduced by Bartels and Reich [BR07] (in the K -theory case) and Bartels and Lück [BL10] (in the L -theory case). In this section we will briefly present their formulation (Subsection 1.1.1), some inheritance properties of the Farrell–Jones Conjecture (Subsection 1.1.2) and a geometric condition implying the Farrell–Jones Conjecture (Subsection 1.1.3). For a general survey on the Farrell–Jones Conjecture (with its various versions and applications) we refer the reader for instance to [LR05], [Lüc10] and [BLR08b]. For an introduction into the methods used to prove the Farrell–Jones Conjecture for various groups we point the reader to [Bar14].

1.1.1 The formulation of the Farrell–Jones Conjecture

The axiomatic formulation of geometric conditions implying the Farrell–Jones Conjecture for a group G introduced by Bartels, Lück and Reich (cf. Subsection 1.1.3) allows us—for the main part of this thesis—to display (or feign) ignorance about the precise nature of most objects that will appear in the conjecture. Thus, we restrict ourselves to stating the conjecture (as formulated in [BR07] and [BL10]) and giving references to the various notions afterwards.

Recall that a *family* $\mathcal{F}$ of subgroups of G is a collection of subgroups of G closed under conjugation and taking subgroups. Examples being the family $\mathcal{ALL}$ of all subgroups of G , the family $\mathcal{FIN}$ of all finite subgroups of G , and most importantly, the family $\mathcal{VCC}$ of all virtually cyclic subgroups of G .

Definition 1.1. (*K-theoretic Farrell–Jones Conjecture relative to $\mathcal{F}$*) A group G satisfies the *K-theoretic Farrell–Jones Conjecture relative to the family of subgroups $\mathcal{F}$* if for all additive G -categories $\mathcal{A}$ the *K-theoretic assembly map*

$$\mathrm{asmb}_n(G, \mathcal{F}, \mathcal{A}) : H_n^G(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \rightarrow H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}) \cong K_n \left(\int_G \mathcal{A} \right),$$

induced by the projection $E_{\mathcal{F}}G \rightarrow \mathrm{pt}$, is an isomorphism for all $n \in \mathbb{Z}$.

Definition 1.2. (*L-theoretic Farrell–Jones Conjecture relative to $\mathcal{F}$*) A group G satisfies the *L-theoretic Farrell–Jones Conjecture relative to the family of subgroups $\mathcal{F}$* if for all additive G -categories $\mathcal{A}$ with involution the *L-theoretic assembly map*

$$\mathrm{asmb}_n(G, \mathcal{F}, \mathcal{A}) : H_n^G(E_{\mathcal{F}}G; \mathbf{L}_{\mathcal{A}}^{-\infty}) \rightarrow H_n^G(\mathrm{pt}; \mathbf{L}_{\mathcal{A}}^{-\infty}) \cong L_n^{\langle -\infty \rangle} \left(\int_G \mathcal{A} \right),$$

induced by the projection $E_{\mathcal{F}}G \rightarrow \mathrm{pt}$, is an isomorphism for all $n \in \mathbb{Z}$.

The following list gives references for and some details on all the objects appearing in the above definitions and is mainly added for the sake of completeness. Only the very last section in this thesis, Section 3.2, briefly assumes familiarity with G -CW-complexes, G -homology theories and the characterization of the space $E_{\mathcal{F}}G$ mentioned earlier, in the introduction.

- An additive G -category is an additive category with a strict right G -action by functors of additive categories. An additive G -category with involution is one in the sense of [BL10, Definition 4.22]. The additive category (with involution) $\int_G \mathcal{A}$ associated to the additive G -category (with involution) $\mathcal{A}$ is constructed [BL10, Section 5 resp. Section 7].
- For the classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$ we refer to [LR05, p. 739] for a definition and a quick overview of its relevant properties. (Extensive information about G -CW-complexes and classifying spaces for families of subgroups can be found in the survey [Lüc05].)
- $H_*^G(-; \mathbf{K}_{\mathcal{A}})$ and $H_*^G(-; \mathbf{L}_{\mathcal{A}}^{-\infty})$ are G -homology theories (see [LR05, p. 739]) obtained via the Davis–Lück construction (see [DL98, Section 4]) from suitable $\mathrm{Or}(G)$ -spectra $\mathbf{K}_{\mathcal{A}}$ and $\mathbf{L}_{\mathcal{A}}^{-\infty}$, respectively (see [BR07, Section 3] and [BL10, Section 9], respectively).
- Finally, $K_n(\int_G \mathcal{A})$ and $L_n^{\langle -\infty \rangle}(\int_G \mathcal{A})$ denote the homotopy groups of the spectra $\mathbf{K}_{\mathcal{A}}(G/G)$ and $\mathbf{L}_{\mathcal{A}}^{-\infty}(G/G)$, respectively. For suitable additive categories (with involution) these groups coincide with the groups $K_n(RG)$ and $L_n^{\langle -\infty \rangle}(RG)$, respectively.

As of now we use the term *Farrell–Jones Conjecture (relative to $\mathcal{F}$)* when talking about both of the above definitions simultaneously. So in particular, and contrary to the existing literature, we do *not* take $\mathcal{VCYC}$ to be the default family.

1.1.2 Inheritance properties

The Farrell–Jones Conjecture described above has various useful inheritance properties (see for instance [BEL08, Section 5] and [BFL14, Section 2.3]). We collect here the ones we will use later on.

Theorem 1.3. ([BFL14, Theorem 2.8]) *Let G be a group that satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{VCYC}(G)$. Let $H \leq G$ be a subgroup. Then H satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to the family $\mathcal{VCYC}(H)$.*

Theorem 1.4 (Transitivity Principle). ([BFL14, Theorem 2.10]) *Let $\mathcal{F} \subset \mathcal{G}$ be two families of subgroups of G . Assume that each group $H \in \mathcal{G}$ satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to the family $\mathcal{F} \cap H := \{K \leq H \mid K \in \mathcal{F}\}$ ¹. Then G satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{F}$ if and only if G satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{G}$.*

Corollary 1.5. *Let G be a group and $\mathcal{F} \subseteq \mathcal{F}'$ be two families of subgroups of G . If G satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{F}$, then G satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{F}'$.*

The sources that give inheritance properties for the Farrell–Jones Conjecture have mostly been interested in the conjecture relative to the family $\mathcal{VCYC}$, and the author could not find the following lemma in the literature.

Lemma 1.6. *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). Let $G = \operatorname{colim}_{i \in I} G_i$, with structure maps $\phi_i : G_i \rightarrow G$, and let $\mathcal{F}$ be a family of subgroups of G . If for all $i \in I$ the group G_i satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\phi_i^* \mathcal{F} := \{H \leq G_i \mid \phi_i(H) \in \mathcal{F}\}$, then G satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{F}$.*

A vague proof. In the L -theoretic case, if $\mathcal{F} = \mathcal{VCYC}$ and $\phi_i^* \mathcal{F}$ is replaced by $\mathcal{VCYC}$ as well ($\mathcal{VCYC}(G_i)$ that is), then this lemma is [BL10, Corollary 0.8]. The proof for arbitrary families and the K -theoretic case work similarly. $\square$

On second glance this argumentation is too vague to be satisfactory for our purposes: The proof of [BL10, Corollary 0.8] indicated in [BL10] requires the families involved to come from a class of groups closed under isomorphisms and taking quotients. An assumption which we want to avoid in the context of group actions on trees. Therefore, a detailed proof of Lemma 1.6 is given in Appendix A.

¹[BFL14] says $K \in \mathcal{G}$ here, which is a misprint. Compare [BEL08, Theorem 4.3], which is referenced in the proof of [BFL14, Theorem 2.10].

1.1.3 A geometric condition implying the Farrell–Jones Conjecture

Along with the proof of the K -theoretic Farrell–Jones Conjecture for hyperbolic groups, Bartels, Lück and Reich showed that the existence of a particular group action $G \curvearrowright X$ admitting intricate G -invariant covers implies the Farrell–Jones Conjecture (cf. [BLR08a, Theorem 1.1]). Since then these geometric conditions have been reformulated, generalised and adapted in several ways (cf. [BL12a, Section 1], [Weg12, Theorem 1.1 and Sections 2 and 3], [Bar16, Definition 0.1 and Theorem 4.3]). In this subsection we will present the formulation that suits us most while mainly following [Bar16].

Definition 1.7. Let $\mathcal{F}$ be a family of subgroups of a group G . A subset $U \subset X$ of a G -space X is called an $\mathcal{F}$ -subset if there is $F \in \mathcal{F}$ such that $gU = U$ for $g \in F$ and $gU \cap U = \emptyset$ for $g \notin F$. A collection $\mathcal{U}$ of subsets of X is said to be G -invariant if $gU \in \mathcal{U}$ for all $g \in G$ and $U \in \mathcal{U}$. An (open) $\mathcal{F}$ -cover of X is a cover of X consisting of (open) $\mathcal{F}$ -subsets.

Definition 1.8. Let $\mathcal{U}$ be a collection of subsets of some space X . The *order of $\mathcal{U}$* is $\leq N$ if each $x \in X$ is contained in at most $N + 1$ members of $\mathcal{U}$. If the collection $\mathcal{U}$ is a cover of X , the order of $\mathcal{U}$ will also be called the *dimension of $\mathcal{U}$* and then is denoted by $\dim \mathcal{U}$.

For the connection of the next definition to amenable actions we refer the reader to [Bar16, Remark 0.4].

Definition 1.9. (cf. [Bar16, Definition 0.1]) Let G be a countable group (with a chosen proper (left-)invariant metric) and let $\mathcal{F}$ be a family of subgroups. An action of G on a space X is $N - \mathcal{F}$ -amenable if for any $\alpha > 0$ there exists an open G -invariant $\mathcal{F}$ -cover $\mathcal{U}_\alpha$ of $G \times X$ (equipped with the diagonal action) with the following properties:

- a) the dimension of $\mathcal{U}_\alpha$ is at most N ;
- b) for all $(g, x) \in G \times X$ there is $U \in \mathcal{U}_\alpha$ with $B_\alpha(g) \times \{x\} \subset U$.

An action is *finitely $\mathcal{F}$ -amenable* if it is $N - \mathcal{F}$ -amenable for some N .

Since we will not construct those types of covers in one go, we introduce the following useful terminology reminiscent of [BLR08a, Assumption 1.4].

Definition 1.10. Let $\alpha > 0$ be given. A collection $\mathcal{U}$ of subsets of $G \times X$ is *wide for Y* , where $Y \subset G \times X$ is a subset, if for all $(g, x) \in Y$ there is $U \in \mathcal{U}$ with $B_\alpha(g) \times \{x\} \subset U$. The collection $\mathcal{U}$ is *wide* if it is wide for $Y = G \times X$.

Definition 1.11. ([BL12a, Definition 1.5]) Let X be a metric space and $N \in \mathbb{N}$. Then X is *controlled N -dominated* if, for every $\varepsilon > 0$, there are a finite CW-complex K_ε of dimension at most N , maps $i : X \rightarrow K_\varepsilon, p : K_\varepsilon \rightarrow X$ and a homotopy $H : X \times [0, 1] \rightarrow X$ between $p \circ i$ and id_X such that for every $x \in X$, the diameter of $\{H(x, t) \mid t \in [0, 1]\}$ is at most ε .

X is *controlled finitely-dominated* if X is controlled N -dominated for some N .

Theorem 1.12. (cf. [Bar16, Theorem 4.3]) *Let G be a group and $\mathcal{F}$ be a family of subgroups. Let $\mathcal{F}_2$ be the family of subgroups of G that contain a group in $\mathcal{F}$ as a subgroup of index ≤ 2 . If G admits a finitely $\mathcal{F}$ -amenable action on a compact contractible controlled finitely-dominated metric space X , then*

- a) *G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}$;*
- b) *G satisfies the L -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}_2$.*

Proof. b) is the conclusion of [BL12a, Theorem 1.1 (ii)]. The assumptions of [BL12a, Theorem 1.1] were phrased as ‘ G is transfer reducible over $\mathcal{F}$ ’ (cf. [BL12a, Definition 1.8]). The notion of transfer reducibility is more general than $N - \mathcal{F}$ -amenability of an action on a suitable space. (Transfer reducibility neither demands a strict action of G on X , nor that X is independent of the given $\alpha > 0$.) Moreover, transfer reducibility demands G -invariant covers for the action of G on $G \times X$ that is given by $h(g, x) := (hg, x)$. Nevertheless, using that $(g, x) \mapsto (g, g^{-1}x)$ is a G -invariant homeomorphism from $G \times X$ with the diagonal action to $G \times X$ with the action described above, it is straightforward to check from the respective definitions that the existence of a finitely $\mathcal{F}$ -amenable action of G on a compact contractible controlled finitely-dominated metric space X implies that G is transfer reducible over $\mathcal{F}$ (cf. the comment behind [BL12a, Definition 1.4]).

a) is the conclusion of [Weg12, Theorem 1.1]. The assumptions of [Weg12, Theorem 1.1] were phrased as ‘ G is strongly transfer reducible over $\mathcal{F}$ ’ (cf. [Weg12, Definition 3.1]). Again, this notion is more general than $N - \mathcal{F}$ -amenability and one can check, by translation of terminology, that the existence of a finitely $\mathcal{F}$ -amenable action of G on a compact contractible controlled finitely-dominated metric space X implies that G is strong transfer reducible over $\mathcal{F}$. $\square$

Thicken open sets

While showing $N - \mathcal{F}$ -amenability, we will occasionally have to thicken sets that are open in a subspace Y to open sets of the ambient space. There is a convenient procedure to do this (cf. [Bar16, Appendix B] and [BL12b, Lemma 4.14]), which we will state here in a formulation tailored to our situation.

Definition 1.13. (cf. [BL12b, Lemma 4.14]) Let Z be a metric space and Y a subspace of Z . For an open subset U of Y let

$$U^+ := \{z \in Z \mid d_Z(z, U) < d_Z(z, Y \setminus U)\}.$$

Lemma 1.14. ([BL12b, Lemma 4.14]) *The subset U^+ is open in Z and $U = U^+ \cap Y$.*

Let X be a metrisable space and G be a group with a left invariant metric. Then the space $G \times X$ equipped with the action ϕ given by $h(g, x) := (hg, x)$ has a G -invariant metric. The G -space $(G \times X, \phi)$ is equivariantly homeomorphic to $G \times X$ with the diagonal action, and hence, $G \times X$ with the diagonal action admits a G -invariant metric as well.

Lemma 1.15. (cf. [Bar16, Lemma B.2]) *Let X and G be as above and let $\mathcal{F}$ be a family of subgroups of G . Let $Y \subset G \times X$ be a G -invariant subspace. Let $\mathcal{U}$ be an open G -invariant $\mathcal{F}$ -cover of Y that is wide for Y (and an implicitly fixed $\alpha > 0$). Then $\mathcal{U}^+ := \{U^+ \mid U \in \mathcal{U}\}$ is a G -invariant collection of open $\mathcal{F}$ -subsets of $G \times X$ that is wide for Y (and the implicitly fixed $\alpha > 0$). Furthermore, the dimension of $\mathcal{U}$ and the order of $\mathcal{U}^+$ agree.*

Proof. It is immediate that $\mathcal{U}^+$ is wide for Y since $U = U^+ \cap Y$ holds. The rest was shown in [Bar16, Lemma B.2]. $\square$

1.2 Acylindrical actions on simplicial trees

The purpose of this section is to introduce the notion of an acylindrical action of a group G on a tree T and describe how one could construct examples for such actions.

For the classical knowledge about actions on trees (Bass-Serre theory)—in particular for the combinatorial definition of a tree as well as the definition of translations length and -axis—we refer the reader to [Ser80]. By assigning length 1 to each edge in a tree T , one can turn T into a metric space (see for instance [BH99, I.1.9 Metric Graphs]). This metric space is a 1-dimensional contractible simplicial complex. Furthermore, it is a (complete) geodesic metric space and is 0-hyperbolic, and hence, it is an $\mathbb{R}$ -tree by definition (see for instance [CHL07, Definition 1.2]). We will rarely use the last fact, but occasionally our references point to results for $\mathbb{R}$ -trees. In the sequel we will always write *simplicial tree* (or *tree*) regardless of whether we think of the purely combinatorial object or the geodesic metric space obtained from it. In particular, we have the notion of a *geodesic* and a *geodesic segment* (i.e. the image of a geodesic) in a simplicial tree. Finally, an action $G \curvearrowright T$ is always understood to come from an action of G on the combinatorial object T , and hence is always simplicial.

Before giving the definition of a k -acylindrical action, we include the following basic lemma, which allows us to assume actions of finitely generated groups on trees to automatically be cocompact. However, note that in general we do not assume our groups to be finitely generated (and thus can not and will not assume the actions $G \curvearrowright T$ to be cocompact).

Lemma 1.16. *Let G be a finitely generated group acting on a simplicial tree T . Then there is a minimal G -invariant subtree $T_{\min}$ of T such that G acts cocompactly on $T_{\min}$.*

Proof. If each element of G fixes a point in T , then—since G is finitely generated—the action $G \curvearrowright T$ has a global fixed point (see [Ser80, Corollary 3, p. 65]) and the claim is obvious. So from now on we assume that there is an element $g \in G$ with translation length > 0 . For each such element we denote its translation axis by C_g . So

$$T_{\min} := \bigcup_{g \in G, \|g\| > 0} C_g$$

is non-empty. It follows from [CM87, p. 577, 1.5, 1.6] that $T_{\min}$ is connected and hence a subtree of T . Since, for any $h, g \in G$, we have $hC_g = C_{hgh^{-1}}$ and it is $\|g\| > 0$ if and only

if $\|gh^{-1}\| > 0$, $T_{\min}$ is G -invariant. It is also minimal G -invariant, since deleting (part of) one (of the) translation axis (axes) destroys G -invariance immediately.

Now let S be a finite generating system for G . Choose $x_0 \in T_{\min}$ and let K be the finite subtree of $T_{\min}$ spanned by $\{x_0, sx_0 \mid s \in S\}$. Define $\hat{T} := \bigcup_{g \in G} gK$. The action of G on $T_{\min}$ restricts to an action of G on $\hat{T}$ and this action is cocompact. We show that $\hat{T}$ is connected and hence a tree: Let z be a point of gK . Write $g = s_1 \dots s_n$. Then $s_1 \dots s_{i-1}K \cap s_1 \dots s_iK \ni s_1 \dots s_ix_0$ for $i = 1, \dots, n$, and thus we have a path from x_0 to z in $\hat{T}$. Since the action of G on $T_{\min}$ was minimal, we conclude $\hat{T} = T_{\min}$. $\square$

The original definition of a k -acylindrical action of a group on a simplicial tree is due to Sela [Sel97, p. 528]. And in recent years Osin established Bowditch's more general notion of an acylindrical action of a group on a path-metric space [Bow08, p. 284] as part of his definition of acylindrically hyperbolic groups [Osi16]. We will however use Delzant's definition of a $(k, \mathcal{FLN})$ -acylindrical action (see [Del99, Definition, p. 1215]), which (in the case of simplicial actions on trees) generalises both Sela's and Bowditch's definition.

Definition 1.17. Let $k \geq 0$. An action $G \curvearrowright T$ on a simplicial tree is *k -acylindrical* if the pointwise stabiliser of any geodesic segment of length k is finite. The action is *uniformly k -acylindrical*, if there is a uniform bound on the cardinality of the pointwise stabiliser of geodesic segments of length k . The action is *strictly k -acylindrical*, if the pointwise stabiliser of any geodesic segment of length k is trivial. An action is *(strictly/uniformly) acylindrical*, if it is (strictly/uniformly) k -acylindrical for some $k \in \mathbb{N}$.

Remark 1.18. In Sela's original definition of k -acylindricity the action was required to be strictly $k + 1$ -acylindrical in the above sense. And Bowditch's definition applied to a simplicial tree is equivalent to a uniformly k -acylindrical action in the above sense.

In the case $k = 0$ the action $G \curvearrowright T$ has finite point stabilisers. Hence, if G is finitely generated, then the existence of a 0-acylindrical action on a simplicial tree implies that G is word hyperbolic (since, by the above lemma, we can assume the action to automatically be cocompact).

In the case $k = 1$ the action $G \curvearrowright T$ has finite edge stabilisers. Since any tree is, in particular, a fine hyperbolic graph in the sense of Bowditch (cf. [Bow08, Definition after Proposition 2.1, p. 11]), if G is finitely generated the existence of a 1-acylindrical action on a simplicial tree with finitely generated vertex stabilisers implies that G is strongly relatively hyperbolic, i.e. relatively hyperbolic in the sense of Bowditch (cf. [Bow08, Definition 2]) (again, cocompactness of the action can be assumed by the above lemma). Examples of 1-acylindrical actions on simplicial trees can be readily given: For groups A and B with an (up to isomorphism) common finite subgroup C , the amalgamated free product $A *_C B$ acts 1-acylindrically on the Bass-Serre tree associated to $A *_C B$.

Plenty of examples for k -acylindrical actions that are not 1-acylindrical can be given using the classical notion of an (almost) malnormal subgroup.

Definition 1.19. A subgroup $H \leq G$ is *malnormal* resp. *almost malnormal* if for all $g \in G \setminus H$ the intersection $H \cap gHg^{-1}$ is trivial resp. finite. We call a subgroup $H \leq G$ *uniformly almost malnormal* if there is a bound $K > 0$ such that for all $g \in G \setminus H$ the intersection $H \cap gHg^{-1}$ has at most K elements.

Lemma 1.20. (cf. [Sel97, p. 528]) *Let $C \leq A$ be (uniformly) almost malnormal and let $C \leq B$. Then the action of $G := A *_C B$ on the Bass-Serre tree associated to G is (uniformly) 3-acylindrical. In particular, if C is infinite, then G does not act 1-acylindrical on its Bass-Serre tree.*

Proof. Let T be the Bass-Serre tree of $A *_C B$ and let σ be a geodesic segment of length 3 in T starting and ending in a vertex. We have to show that the pointwise stabiliser of σ is finite (resp. is bounded by some uniform constant).

By the construction of T (see [Ser80, p. 33]), without loss of generality the vertices on σ are of the form $gA, gB, gbA, gbaB$ for some $g \in G, a \in A \setminus C, b \in B \setminus C$. Thus, the edges along $b^{-1}g^{-1}\sigma$ are of the form $b^{-1}C, C, aC$. The pointwise stabiliser of $(gb)^{-1}\sigma$ (and hence of σ) has cardinality at most $|C \cap aCa^{-1}|$. If $C \leq A$ is almost malnormal, $|C \cap aCa^{-1}|$ is finite. And if $C \leq A$ is uniformly almost malnormal, $|C \cap aCa^{-1}|$ is bounded by some constant independent of a . $\square$

Similar statements can be formulated for HNN-extensions and graphs of groups using the notion of an almost malnormal collection of subgroups.

There are plenty of examples for (almost) malnormal subgroups (for an overview on results giving such group see for instance [dlHW14]) and only a few of them will be mentioned here. First, as one particular source for examples of malnormal subgroups we cite the following theorem by Kapovich.

Theorem 1.21. ([Kap99, Theorem C]) *Let G be a torsion-free word hyperbolic group and let Γ be a non-elementary (i.e. not virtually cyclic) subgroup of G . Then there exists a subgroup H of Γ such that H is a free group of rank two which is (quasiconvex and) malnormal in G .*

If one does not fix the ambient group in advance, then any group H can be made into a malnormal subgroup of some ambient group: just consider $H \leq H * H'$ for any non-trivial H' . However, any finitely generated torsion free group can also be made into a malnormal subgroup in a non-trivial way as the following theorem by Osin says.

Theorem 1.22. ([dlHW14, Theorem 15]) *Let H be a finitely generated torsion free group H and $Q \leq H$ be finitely generated. Then there is a group N , a split extension $G = N \rtimes H$ and an element $z \in N$, such that H is malnormal in G and $C_G(z)/C_N(z) \cong Q$.*

An interesting class of uniformly almost malnormal subgroups are peripheral groups in relatively hyperbolic groups.

Lemma 1.23. ([DS08, Lemma 4.20]) *Let G be an infinite finitely generated group that is relatively hyperbolic with respect to a finite collection of finitely generated subgroups $\{H_1, \dots, H_n\}$. Then the collection $\{H_1, \dots, H_n\}$ is uniformly almost malnormal. In particular, each H_i is uniformly almost malnormal in G .*

Finally, we include an example of a non-uniformly almost malnormal subgroup. The idea to use an iterated HNN-extension to give such an example was suggested to the author by Rémi Coulon.

Example 1.24. Let $G_0 := \prod_{i \in \mathbb{Z}} \mathbb{Z}/2$. For $j > 0$ let

$$H_j := \prod_{i \leq 0} \{1\} \times (\mathbb{Z}/2)^j \times \prod_{i > j} \{1\} \leq G_0$$

and similarly for $j < 0$ let

$$H_j := \prod_{i \leq j} \{1\} \times (\mathbb{Z}/2)^{|j|} \times \prod_{i \geq 0} \{1\} \leq G_0.$$

Let ϕ_j denote the obvious isomorphism from $(\mathbb{Z}/2)^{|j|}$ to H_j . We iteratively define the group G_n to be the HNN-extension of G_{n-1} over $(\mathbb{Z}/2)^n$ with respect to the two inclusions ϕ_n and ϕ_{-n} . Let $G := \text{colim}_{n \in \mathbb{N}} G_n$. Thus, if we write $G_0 = \langle S_{G_0} \mid R_{G_0} \rangle$, a representation of G is given as follows.

$$G = \langle S_{G_0}, t_n, n \in \mathbb{N} \mid R_{G_0}, t_n \phi_n(h) t_n^{-1} = \phi_{-n}(h), h \in (\mathbb{Z}/2)^n, n \in \mathbb{N} \rangle.$$

An iterative use of Britton's lemma [LS77, Britton's lemma, p. 181] shows that $G_0 \leq G$ is almost malnormal, i.e. that all intersections of the form $gG_0g^{-1} \cap G_0$ are finite if $g \notin G_0$. And since for all t_n the intersection $t_n G_0 t_n^{-1} \cap G_0$ contains $t_n H_n t_n^{-1} = H_{-n}$, the subgroup G_0 is not uniformly almost malnormal in G . Using Lemma 1.20 for any group G' that contains G_0 (for instance $G' = \text{colim}_{n \in \mathbb{N}} \text{SL}(n, \mathbb{Z})$) then gives a 3-acylindrical action which is not uniform.

It is worth noting that not all acylindrical actions arise in this fashion from (almost) malnormal subgroups. As can be seen (with some translation of terminology) from the type of example given in [KS71, p. 946 and p. 951], there are amalgamated free products $A *_C B$ that act 3-acylindrically on their associated Bass-Serre tree, with C not being almost malnormal in A or B .

1.3 The observers' topology for trees

For proper geodesic metric hyperbolic spaces (so in particular for locally finite trees T) adding the Gromov boundary results in a compact metrisable space (see [BH99, III.H.3, in particular 3.7, 3.18 (4)]). Since most trees in this thesis will not be locally finite, changing the topology on T itself becomes necessary.

Bowditch gave a compactification of (the vertex set of) a fine hyperbolic graph [Bow12, Section 8, p. 51ff] via the so-called observers' topology. A hyperbolic graph is fine, if for all $C > 0$ it only contains finitely many arcs of length $\leq C$ between any two given vertices. So in particular a tree is fine. This section gives the necessary background on the observers' topology for simplicial trees and collects its properties needed throughout the thesis.

We start by setting up some general notation. Let T be a simplicial tree (viewed as a metric space). A *geodesic ray* in T is an isometry $\gamma : [0, \infty) \rightarrow T$. Two geodesic rays are equivalent if the Hausdorff distance between their images is finite. Since T is a tree, this is the case if and only if the intersection of their images is unbounded. The *geodesic boundary of T* , i.e. the set of equivalence classes of geodesic rays in T , is denoted by ∂T . Since T is a tree—whether it is a proper metric space or not—the geodesic boundary of T is homeomorphic to the Gromov boundary $\partial_s T$ defined via sequences convergent at infinity (see [Gro87, 1.8]). Moreover, note that given a point $[\gamma] \in \partial T$ and a point $b \in T$ we can always find a unique representative γ' of $[\gamma]$ which starts at b . So whenever convenient we will view ∂T as the set of all geodesic rays in T emanating from a given point $b \in T$.

A *bi-infinite geodesic ray* in T is an isometry $\rho : \mathbb{R} \rightarrow T$. Note that for $[\gamma] \neq [\gamma'] \in \partial T$ there is a unique bi-infinite geodesic ray *from* $[\gamma]$ *to* $[\gamma']$, i.e. there is a unique $\rho : \mathbb{R} \rightarrow T$ such that $[\gamma] = [\rho|_{[0, \infty)}]$ and $[\gamma'] = [\rho|_{(-\infty, 0]} \circ (t \mapsto -t)]$. A *generalised geodesic* in T is a continuous map $\gamma : [0, \infty) \rightarrow T$ for which there is an $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $\gamma|_{[0, R)}$ is an isometry and $\gamma|_{[R, \infty)}$ is constant. Since T is a tree, after fixing a basepoint $b \in T$ there is a unique generalised geodesic in T from b to a given $x \in T \cup \partial T$. This allows us to think of points in T as generalised geodesics, which will occasionally be serviceable for us.

The term *geodesic* will—besides for actual finite geodesics in T —also be used for geodesic rays as well as bi-infinite geodesic rays in T .

Notation 1.25. For two points x, y in $T \cup \partial T$ we denote both the geodesic from x to y and its image by $[x, y]$. If $x \in T$, we will occasionally use the same notation for the unique generalised geodesic from x to y . Moreover, we use $(x, y]$ and $[x, y)$ for $[x, y] \setminus \{x\}$ and $[x, y] \setminus \{y\}$, respectively (provided $x \in T$ and $y \in T$, respectively).

For a tree T let $V(T)$ denote the set of vertices of T and let $V_\infty(T) \subset V(T)$ be the set of vertices of infinite degree (valency). Bowditch's construction of the observers' topology amounts to compactifying $\Delta(T) := V_\infty(T) \cup \partial T$. We will need to use a compactification—via the observers' topology—of all of $T \cup \partial T$ instead. Written in the more general language of $\mathbb{R}$ -trees this can for instance be found in [CHL07, Section 1]. In particular, compactness of the resulting space is shown in [CHL07, Proposition 1.13]. As the nature of the observers' topology is central to the whole of Chapter 2, we will nevertheless describe the observers' topology for simplicial trees in detail.

Let $z \in T \cup \partial T$ and $A \subset T$ finite. Define

$$M(z, A) := \{y \in T \cup \partial T \mid A \cap [z, y] = \emptyset\}. \quad (1.1)$$

It is both straightforward and instructional to verify

$$M(z, A) = M(x, A) \text{ if } x \in M(z, A) \quad (1.2)$$

and $M(z, A) \cap M(z, B) = M(z, A \cup B)$ with the help of Figure 1.1. So the collection

$$\{M(z, A) \mid z \in T \cup \partial T, A \subset T \text{ finite}\} = \{M(z, A) \mid z \in T, A \subset T \text{ finite}\}$$

forms a basis for a topology on $T \cup \partial T$ and the collection $\{M(z, A) \mid A \subset T \text{ finite}\}$ forms an open neighbourhood basis for z in this topology. We will abbreviate $M(z, \{a\})$ to $M(z, a)$. Moreover, when talking about $\Delta(T) = V_\infty(T) \cup \partial T$ we will also use the notation $M(z, A)$ as a shorthand for $M(z, A) \cap \Delta(T)$ whenever this is convenient.

Figure 1.1: $M(z, A) \cap M(z, B) = M(z, A \cup B)$ and $M(z, A) = M(x, A)$ for $x \in M(z, A)$.

Definition 1.26. The topology on $T \cup \partial T$ given by the basis

$$\{M(z, A) \mid z \in T \cup \partial T, A \subset T \text{ finite}\}$$

(with $M(z, A)$ as in (1.1)) is called the *observers' topology*. The resulting space is the *Bowditch compactification* of T and will be denoted by $\overline{T}^{obs}$. The *Bowditch boundary* of T is the subspace $\Delta(T) = V_\infty(T) \cup \partial T \subset \overline{T}^{obs}$.

The above naming is justified since it is compatible with Bowditch's construction, i.e. the subspace topology on $\Delta(T) \subset \overline{T}^{obs}$ is the topology constructed by Bowditch in [Bow12, Section 8, p. 51ff].

Remark 1.27. On any finite subtree $S \subset T$ the restriction of the observers' topology to S coincides with the path-metric topology (i.e. the topology obtained by assigning each edge length 1, thereby turning a tree into a metric space). So if a sequence x_n in T is contained in a finite subtree $S \subset T$ (i.e. after fixing a basepoint b the union of the $[b, x_n]$ is a finite subtree of T), then x_n converges to x in the observers' topology if and only if x_n converges to x in the path-metric topology.

To further illustrate the nature of the observers' topology we include the following lemma.

Lemma 1.28. *Let x_n be a sequence in $\overline{T}^{obs}$ not contained in a finite subtree of T .*

- a) $x_n \rightarrow x \in \partial T$ if and only if (after fixing a basepoint $b \in T$) for all $C \geq 0$ there is an $n_0 \in \mathbb{N}$ such that $[b, x]_{[0, C]} = [b, x_n]_{[0, C]}$ for all $n \geq n_0$. (Since T is a tree, this is equivalent to the sequence of generalised geodesics $[b, x_n]$ converging uniformly on compacta to the unique geodesic ray $[b, x]$).
- b) If $x_n \rightarrow x \in V_\infty(T)$, then the (union of the) generalised geodesics $[x, x_n]$ intersect infinitely many of the edges incident to x .
- c) Let $x \in V_\infty(T)$. If the (union of the) generalised geodesics $[x, x_n]$ intersect infinitely many of the edges incident to x , then x_n subconverges to x (i.e. there is a subsequence x_i of x_n converging to x).

Proof. a) As can be verified best from Figure 1.2, an open neighbourhood basis at $x \in \partial T$ is given by the $M(x, b_i)$ where b_i denotes the i th vertex on $[b, x]$. Then the claim is evident.

Figure 1.2: If $x \in \partial T$, then any $M(x, A)$ contains an $M(x, b_i)$ for some b_i on $[b, x]$.

b) If the generalised geodesics $[x, x_n]$ intersect only finitely many edges $e_1, \dots, e_k$ incident to x , then taking A as the set of midpoints of the e_i gives an open set $M(x, A)$ containing x . If $x_n \rightarrow x$, then $M(x, A)$ had to contain almost all of the x_n . Hence, almost all of the x_n had to lie on the $e_1, \dots, e_k$. Since the sequence x_n is, by assumption, not contained in a finite subtree of T , this is a contradiction.

c) If the generalised geodesics $[x, x_n]$ intersect infinitely many of the edges incident to x , we can choose for each such edge e_i one x_i with $e_i \cap [x, x_i] \neq \emptyset$. Then each $M(x, A)$ has to contain almost all of the x_i , since A is finite. So $x_i \rightarrow x$. $\square$

The rest of this subsection is mainly devoted to showing that $\overline{T}^{obs}$ is a compact contractible controlled finitely-dominated metric space. Although in the general setting of separable $\mathbb{R}$ -trees compactness of $\overline{T}^{obs}$ is shown in [CHL07, Proposition 1.3], we include a detailed proof for simplicial trees written in our notation as a kind of warm-up.

Lemma 1.29. *The space $\overline{T}^{obs}$ is Hausdorff. Moreover, it is regular, i.e. for each point $x \in \overline{T}^{obs}$ and $S \subset \overline{T}^{obs}$ closed with $x \notin S$, there are disjoint open neighbourhoods U and V of x and S , respectively.*

Proof. If $x \neq y$, then choosing any point a in the interior of $[x, y]$ results in disjoint open sets $M(x, a)$ and $M(y, a)$ that contain x and y , respectively. To show that $\overline{T}^{obs}$ is regular let $x \in \overline{T}^{obs}$ and $S \subset X$ be closed with $x \notin S$. Then $\overline{T}^{obs} \setminus S$ is open and hence can be written as some union $\bigcup_{i \in I} M(z_i, A_i)$ for suitable finite A_i and (without loss of generality) $z_i \in T$. Say $x \in M(z_0, A_0) = M(x, A_0)$. Now for all $a \in A_0$, choose a point a' in the interior of $[x, a]$, another point a'' in the interior of $[a', a]$ and a point z' in the interior of $[a'', a]$. By construction $x \in M(x, A')$ with A' being the finite set of all a' . And $\bigcup_{a \in A_0} M(z', a'')$ contains $\overline{T}^{obs} \setminus M(x, A_0)$, which contains S . $\square$

Lemma 1.30. *If T is countable (i.e. has only countably many edges), then $\overline{T}^{obs}$ is second-countable and metrisable. In particular, $\overline{T}^{obs}$ and $\Delta(T)$ are separable.*

Proof. If T is countable, then T is separable. Choose a countable dense subset Q of T which contains all vertices of T . Then

$$\{M(z, A) \mid z \in Q, A \subset Q \text{ finite}\}$$

is countable and still a basis for the observers' topology, as $M(x, A) = M(z, A)$ holds for all $z \in M(x, A)$. So $\overline{T}^{obs}$ is second-countable. Since, by the above lemma, $\overline{T}^{obs}$ is also Hausdorff and regular, the space $\overline{T}^{obs}$ is metrisable by the Urysohn Metrization Theorem (see for instance [Kel55, 16 Metrization Theorem, p. 125]). Finally, for metric spaces being second-countable and being separable are equivalent properties. $\square$

Lemma 1.31. *If T is countable, then the space $\overline{T}^{obs}$ is compact and the subspaces $\Delta(T)$ and $\Delta_+(T) := V(T) \cup \partial T$ are compact as well.*

Proof. Since $\overline{T}^{obs}$ is metrisable, it suffices to show that it is sequentially compact. Let x_n be a sequence in $\overline{T}^{obs}$. If x_n has a subsequence contained in a finite subtree S of T , then, by compactness of S , there is a convergent subsequence of x_n . So assume from now on that x_n is not contained in any finite subtree of T .

Let b_0 be any vertex of T . If the (union of the) generalised geodesics $[b_0, x_n]$ intersect infinitely many edges incident to b_0 (in particular $b_0 \in V_\infty(T)$ has to hold), then, by Lemma 1.28 c), there is a subsequence of x_n converging to b_0 . If this is not the case, then there is (at least) one edge e_0 incident to b_0 and a subsequence $x_i^{(1)}$ of x_n such that all $[b_0, x_i^{(1)}]$ contain e_0 . Set b_1 as the other vertex incident to e_0 . Iterating this argument either terminates after finitely many steps, and thus gives a subsequence of the x_n converging to b_t for some $t \in \mathbb{N}$. Or—if the iteration does not terminate—we obtain a geodesic ray y given by the edges $e_0, e_1, \dots, e_j, \dots$, where e_j is the edge chosen in the $(j+1)$ st iteration step. Moreover, we have subsequences $x_i^{(1)}, \dots, x_i^{(j)}, x_i^{(j+1)}, \dots$ of x_n , where $x_i^{(j+1)}$ is a

subsequence of $x_i^{(j)}$ and the restriction $[b, x_i^{(k)}]_{[0,j]}$ is equal to $[b, x_i^{(j)}]$ for all $k \geq j$. Choose y_j arbitrarily from $x_i^{(j)}$. The sequence y_j is not contained in a finite subtree of T and hence converges to $y \in \partial T$ by Lemma 1.28 a). Finally, both $\Delta(T)$ and $\Delta_+(T)$ are closed subspaces of $\bar{T}^{obs}$. $\square$

Definition 1.32. Let X be a regular space. The *small inductive dimension* $\text{ind } X$ is $\leq n$ if for all $x \in X$ and all neighbourhoods $V \subset X$ of x there is an open set $U \subset V$ such that $x \in U \subset V$ and $\text{ind } \partial U \leq n - 1$. The small inductive dimension of the empty set is defined as $\text{ind } \emptyset := -1$.

For a separable and metrisable space, the small inductive dimension coincides with its covering dimension (and the strong inductive dimension) [Nag65, Theorem IV.1, p. 90]. So from now on we will only speak of the dimension of a space.

Lemma 1.33. *The space $\Delta(T)$ has dimension 0.*

Proof. By definition, $\mathcal{B} := \{M(z, A) \mid z \in T \cup \partial T, A \subset T \text{ finite}\}$ is a basis for the topology of $\bar{T}^{obs}$. Now let $z \in \Delta(T)$ and V be a neighbourhood of z in $\Delta(T)$. Then there is a set $M(z, A) \in \mathcal{B}$ with $z \in M(z, A) \cap \Delta(T) =: U$, and hence U is open in $\Delta(T)$. We have to show that the boundary of U (in $\Delta(T)$) is empty.

Let $x_n \rightarrow x$ be a convergent sequence in $\Delta(T)$ with $x_n \in U$ for all $n \in \mathbb{N}$. If the x_n are contained in a finite subtree S of T , then the sequence must be constant (since $x_n \in \Delta(T)$) and $x \in U$ follows. So assume from now on that the sequence is not contained in a finite subtree of T . If $x \notin U$, then there must be an $a \in A$ lying in the interior of $[z, x]$. First, assume that $x \in \partial T$ and take $b = z$ as a basepoint. Then the geodesics $[b, x_n]$ must all be contained in $M(z, A)$. So, in particular, they can not contain a . Since $x_n \rightarrow x$, this contradicts Lemma 1.28 a). Now, if $x \in V_\infty(T)$, take $b = x$ as a basepoint. Then the geodesics $[b, x_n]$ must all contain the point a , as $x_n \in U$. Since $x_n \rightarrow x$, this contradicts Lemma 1.28 b). $\square$

Remark 1.34. Similarly to the above proof one can show that the boundary of $M(z, A)$ in $\bar{T}^{obs}$ is equal to A and is hence finite and 0-dimensional. Therefore, the space $\bar{T}^{obs}$ has dimension 1.

We proceed to show that $\bar{T}^{obs}$ is controlled 1-dominated (see Definition 1.11). We split the proof into two lemmata.

Lemma 1.35. *Let d be a metric on $\bar{T}^{obs}$ compatible with the observers' topology. For every $\varepsilon > 0$ there is a finite subtree $K_\varepsilon \subset T$ and a continuous projection $p : \bar{T}^{obs} \rightarrow K_\varepsilon$ such that $d(x, p(x)) < \varepsilon$ for all $x \in \bar{T}^{obs}$.*

Proof. For this proof we denote the open ball of radius r around $x \in \bar{T}^{obs}$ with respect to d by $B_r(x)$. Certainly,

$$\{B_{\varepsilon/2}(x) \mid x \in \bar{T}^{obs}\}$$

is an open cover of $\overline{T}^{obs}$. Since the $M(x, A)$ with $x \in T \cup \partial T$ and A a finite subset of T form an open neighbourhood basis at x for the observers' topology, there is for each $B_{\varepsilon/2}(x)$ an open set $M(x, A_x) \subset B_{\varepsilon/2}(x)$. So the collection

$$\{M(x, A_x) \mid x \in \overline{T}^{obs}\}$$

is an open cover of $\overline{T}^{obs}$ as well. Since $\overline{T}^{obs}$ is compact, we can find a finite subcover $M(x_1, A_1), \dots, M(x_l, A_l)$. Without loss of generality we can assume that the x_i lie in $V(T)$ (see (1.2)). Let K_ε be the finite subtree of T spanned by $x_1, \dots, x_l$. We define $p : \overline{T}^{obs} \rightarrow K_\varepsilon$ as follows: Fix some point $b \in K_\varepsilon$ and define $p : \overline{T}^{obs} \rightarrow K_\varepsilon$ by sending x to the last vertex on $[b, x]$ that is contained in K_ε .

To show $d(x, p(x)) < \varepsilon$, we argue as follows: For $x \in K_\varepsilon$ we have $d(x, p(x)) = 0$, since $p|_{K_\varepsilon} = \text{id}_{K_\varepsilon}$. For any point $x \notin K_\varepsilon$ there is an x_{i_x} such that $x \in M(x_i, A_i)$. Since K_ε is the tree spanned by $x_1, \dots, x_l$, the point $p(x)$ lies on $[x, x_{i_x}]$, and hence $p(x) \in M(x_i, A_i)$. Therefore, $d(p(x), x) < \varepsilon$. It remains to show that p is continuous. Let $\tilde{\varepsilon} > 0$. For a given $x \in \overline{T}^{obs}$ we will find $\delta > 0$ such that $p(B_\delta(x)) \subset B_{\tilde{\varepsilon}}(p(x))$.

First let $x \in K_\varepsilon$ and, therefore, $x = p(x)$. If x lies in the interior of an edge, then we can find $\delta \leq \tilde{\varepsilon}$ such that $B_\delta(x)$ also is contained in the interior of that edge of K_ε . Thus, $p(B_\delta(x)) = B_\delta(x) \subset B_{\tilde{\varepsilon}}(x)$. If x is a vertex of K_ε , then we consider the open set $M := B_{\tilde{\varepsilon}}(x) \cap M(x, A)$ where $A \subset T$ consists of the midpoints of the edges incident to x in K_ε . That is, we cut off the set $B_{\tilde{\varepsilon}}(x)$ to an open set M such that no other vertex of K_ε is contained in M . Since M is an open neighbourhood of x , there is a $\delta \leq \tilde{\varepsilon}$ such that $B_\delta(x) \subset M$. Apart from x , $B_\delta(x)$ contains only vertices outside of K_ε . Thus, every geodesic starting from b and ending in a point in $B_\delta(x) \setminus K_\varepsilon$ has to contain x as its last shared vertex with K_ε . Hence, for all $y \in B_\delta(x) \setminus K_\varepsilon$ we have $p(y) = x$. Together with $\delta \leq \tilde{\varepsilon}$ it follows that $p(B_\delta(x)) \subset B_{\tilde{\varepsilon}}(p(x))$.

If $x \notin K_\varepsilon$, then $M(x, p(x))$ is a non-empty open set containing x . Thus, there is an open ball $B_\delta(x) \subset M(x, p(x))$. Furthermore, $b \notin M(x, p(x))$. Thus, any geodesic $[b, y]$ starting from b and ending in a point y of $M(x, p(x))$ has to contain $p(x)$. By the definition of p and $M(x, p(x))$, for any $y \in M(x, p(x))$ the vertex $p(x)$ must also be the last shared vertex of $[b, y]$ and K_ε . Hence, $p(y) = p(x)$ for all $y \in M(x, p(x))$. Thus, $p(B_\delta(x)) = \{p(x)\} \subset B_{\tilde{\varepsilon}}(p(x))$. $\square$

Lemma 1.36. *The space $\overline{T}^{obs}$ is controlled 1-dominated (by the K_ε from Lemma 1.35). In particular, $\overline{T}^{obs}$ is contractible.*

Proof. Let d be a metric on $\overline{T}^{obs}$ compatible with the observers' topology. Again, for this proof denote the open ball of radius r around $x \in \overline{T}^{obs}$ with respect to d by $B_r(x)$. Let $\varepsilon > 0$ and let $p : \overline{T}^{obs} \rightarrow K_\varepsilon$ be as in the previous lemma, i.e. we have a finite subtree K_ε of T , fixed a point $b \in K_\varepsilon$ and defined $p(x)$ as the last vertex on $[b, x]$ that also lies in K_ε . We now have to show that there is a homotopy $H : \overline{T}^{obs} \times [0, 1] \rightarrow \overline{T}^{obs}$ between

$\text{id}_{\overline{T}^{obs}}$ and $i \circ p : \overline{T}^{obs} \rightarrow K_\varepsilon \rightarrow \overline{T}^{obs}$, where $i : K_\varepsilon \rightarrow \overline{T}^{obs}$ is the inclusion, such that the set $\{H(x, t) \mid t \in I\}$ has diameter $\leq 2\varepsilon$ for all $x \in \overline{T}^{obs}$.

For $x \in \overline{T}^{obs}$ let $\gamma_{p(x), x} : [0, \infty) \rightarrow T$ be the unique generalised geodesic in T from $p(x)$ to x . We define

$$H(x, t) = \gamma_{p(x), x} \left(\frac{t}{1-t} \right)$$

where for $t = 1$ and $x \in T$ this is to be read as $H(x, 1) = x$, and for $t = 1$ and $x \in \partial T$ this is to be read as $H(x, 1) = [\gamma_{p(x), x}]$. So $H(-, 0) = p$ and $H(-, 1) = \text{id}_{\overline{T}^{obs}}$. Furthermore, note that the set $\{H(x, t) \mid t \in I\}$ is exactly the image of the generalised geodesic $\gamma_{p(x), x}$. Any point on $\gamma_{p(x), x}$ is mapped by p to $p(x)$. Thus, this set has diameter $\leq 2\varepsilon$ by the previous lemma. It remains to show that H is continuous. For this, note first that for $t < 1$ the map $H(-, t)$ is a projection of $\overline{T}^{obs}$ onto the set

$$B_{t \cdot (1-t)^{-1}}^{d_T}(K_\varepsilon) := \left\{ z \in \overline{T}^{obs} \mid d_T(z, K_\varepsilon) \leq \frac{t}{1-t} \right\},$$

where d_T denotes the path-metric on T . Thus, everything inside this set is fixed pointwise. We proceed by a few case distinctions:

Case 1: (cf. Figure 1.3) Let x with $p(x) = x$ be given. Then $H(x, t) = x = p(x)$ for all t . Let $\tilde{\varepsilon} > 0$. Take an open set $M(x, A)$ inside of $B_{\tilde{\varepsilon}}(H(x, t)) = B_{\tilde{\varepsilon}}(x)$ such that $M(x, A)$ does not contain any other vertex of K_ε apart from x itself. Further let $B_\delta(x)$ be any open ball inside of $M(x, A)$. Then for any $z \in B_\delta(x)$ the geodesic $[x, z] = [p(x), z]$ does contain x , but no other vertex of K_ε . In particular, if $z \notin K_\varepsilon$, then $p(z) = p(x)$ for all $z \in B_\delta(x)$. Thus, by the definition of H , the point $H(z, t)$ lies on $[p(x), z]$ for all $z \in B_\delta(x)$. Since any such $[p(x), z]$ is contained in $M(x, A)$, and hence in $B_{\tilde{\varepsilon}}(x)$, it follows that $H(B_\delta(x) \times I) \subseteq B_{\tilde{\varepsilon}}(x)$.

Case 2: (cf. Figure 1.4) Let (x, t) with $p(x) \neq x$, but $H(x, t) = x$ be given. Let $\tilde{\varepsilon} > 0$. Take an open set $M(x, A)$ inside of $B_{\tilde{\varepsilon}}(H(x, t)) = B_{\tilde{\varepsilon}}(x)$. Now choose a point y in the interior of $[x, p(x)]$ and let $B_\delta(x)$ be an open ball inside of $M(x, A \cup \{y\})$. Since $H(x, t) = x$, we find a small s such that the points $H(x, t')$ for t' in $[t-s, t+s]$ lie on $[x, y]$ (if $t = 1$, then adjust the interval $[t-s, t+s]$ accordingly). Now for any $z \in B_\delta(x)$ and $t' \in [t-s, t+s]$ the point $H(z, t')$ lies on $(y, z]$ and this geodesic is contained in $M(x, A \cup \{y\})$, which is contained in $B_{\tilde{\varepsilon}}(x)$.

Case 3: (cf. Figure 1.5) Let (x, t) with $p(x) \neq x$ and $H(x, t) \neq x$ be given. Since $H(x, t) \neq x$, we can choose y in the interior of $[x, H(x, t)]$ and s so small that the points $H(x, t')$ for t' in $[t-s, t+s]$ lie in $M(H(x, t), y)$ (if $t = 0$ then adjust the interval $[t-s, t+s]$ accordingly). Let $\tilde{\varepsilon} > 0$. If necessary, decrease s further such that the points $H(x, t')$ for t' in $[t-s, t+s]$ lie in $M(H(x, t), y) \cap B_{\tilde{\varepsilon}}(H(x, t))$. Choose any open ball $B_\delta(x)$ inside of $M(x, y)$. Now for all $z \in B_\delta(x)$ the point $H(z \times [t-s, t+s])$ lies on the geodesic $(H(x, t-s), H(x, t+s)) \subset B_{\tilde{\varepsilon}}(H(x, t))$.

Since K_ε is finite for all $\varepsilon > 0$, any one of the K_ε suffices to show that $\overline{T}^{obs}$ is contractible. $\square$

Figure 1.3: H is continuous in any point (x, t) with $x = p(x)$. Figure 1.4: H is continuous in any point (x, t) with $x \neq p(x)$, but $x = H(x, t)$. Figure 1.5: H is continuous in any point (x, t) with $x \neq p(x)$ and $H(x, t) \neq x$.

Chapter 2

Acyindrical actions induce finitely $\mathcal{F}$ -amenable actions on the Bowditch boundary

The main work in proving Theorem 3.1 lies in showing that, given an acylindrical action of a countable group G on a countable tree T , the action of G on $\Delta(T)$ is finitely $\mathcal{F}$ -amenable. This chapter deals exclusively with this issue by providing the theorem given below.

Notation 2.1. Let $G \curvearrowright T$ be an action of a group on a tree. We denote by $\mathcal{F}_T$ the family of subgroups $\mathcal{F}_T := \{H \leq G \mid \exists x \in T : H \leq G_x\}$ and by $\mathcal{F}_\partial$ the family of subgroups $\mathcal{F}_\partial := \{H \leq G \mid \exists (x, y) \in \partial T \times \partial T \setminus \text{diag} : H \leq G_x \cap G_y\}$. Here G_x denotes the stabiliser of a point $x \in T \cup \partial T$.

Theorem 2.2. *Let G be a countable group and T be a countable tree. Let $G \curvearrowright T$ be a (not necessarily strictly or uniformly) acylindrical action and let $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_\partial$ be as above. Then the action $G \curvearrowright \Delta(T)$ of G on the Bowditch boundary of T is finitely $\mathcal{F}$ -amenable.*

We briefly recall the overall structure of the proof of Theorem 2.2 from the introduction: First, in Section 2.1 we will define a proper G -invariant metric on the set of all geodesic segments of length k in T . With this metric, we will introduce a notion of Θ -small geodesics in Definition 2.4. Then, similarly to the proof of [Bar16, Theorem 3.1], we use this notion to divide—for each $\alpha > 0$ —the space $G \times \Delta(T)$ into two G -invariant subsets. The first subset consists of all pairs $(g, \xi) \in G \times \Delta(T)$ such that $g^{-1}\xi \in \partial T$ can be reached from a given starting point by a $\Theta(\alpha)$ -small geodesic ray (cf. Notation 2.5). In Sections 2.5–2.7 we will utilise the long thin covers for coarse flow spaces provided by [Bar16, Theorem 1.1] to find a wide cover for this first subset. The second subset, the rest, will be covered using ad hoc techniques developed in Section 2.3 and Section 2.4.

The whole chapter is written under the assumptions in Theorem 2.2, i.e. G and T are countable, we have an action $G \curvearrowright T$ that is k -acylindrical for some k and $\mathcal{F}$ denotes $\mathcal{F}_T \cup \mathcal{F}_\partial$ as in Notation 2.1.

If $G \curvearrowright T$ has a global fixed point, the theorem holds trivially, since then $\mathcal{F} = \mathcal{ALL}$ and $\mathcal{U} := \{G \times \Delta(T)\}$ would be an open $\mathcal{F}$ -cover of dimension 0 that is wide (independent $\alpha > 0$). Thus, we implicitly assume $G \curvearrowright T$ not to have a global fixed point from now on. Moreover and without loss of generality we can assume $k \geq 1$ (which spares us some notational difficulties we would otherwise encounter when trying to treat the cases $k = 0$ and $k \geq 1$ simultaneously).

If $G \curvearrowright T$ is k -acylindrical, then the action of G on the barycentric subdivision T' of T is $2k$ -acylindrical. Furthermore, $\Delta(T) = V_\infty(T) \cup \partial T = \Delta(T')$. Therefore, we can assume without loss of generality that the set $V_0(T) := V(T) \setminus V_\infty(T)$ of vertices of finite degree is unbounded (with respect to the path-metric d_T on T). Moreover, if convenient, we can assume that k is even. The technical importance of those two seemingly artificial assumptions will become clear later on.

2.1 A proper G -invariant metric on $E^k(T)$

In this section we construct a G -invariant proper metric on the set $E^k(T)$ of all geodesic segments of length k in T whose endpoints are vertices of T . This metric will be used in the next section to define the notion of a Θ -small geodesic.

Abels, Manoussos and Noskov showed that for a locally compact group H acting properly on a locally compact, σ -compact metrisable space X , there is an H -invariant proper metric on X compatible with the topology on X [AMN11, Theorem 1.1]. Since $G \curvearrowright T$ is k -acylindrical, the action of G on $E^k(T)$ has finite point stabilisers. Thus, the existence of a G -invariant proper metric on $E^k(T)$ follows directly from [AMN11, Theorem 1.1]. However, the action of G on $E^k(T)$ is, in general, not cocompact and we have to compensate for this by constructing a metric that is ‘cocompact along small geodesics’. One can obtain such a metric by simply making certain choices in the proof of [AMN11, Theorem 1.1] more carefully. In favour of readability we will nevertheless present the whole construction of Abels, Manoussos and Noskov [AMN11] tailored to discrete groups and spaces and, while doing so, cleanly weave our specific additional property into the proof of existence.

Lemma 2.3. *Let G be a countable group and X be a countable discrete space. Furthermore, let $G \curvearrowright X$ have finite point stabilisers. Let $\pi : X \rightarrow G \backslash X$ be the canonical projection and let $F = \{x_0, x_1, \dots\}$ be set of representatives for the orbits of the action $G \curvearrowright X$. Then there is a G -invariant proper metric d_X on X compatible with the discrete topology such that the following holds: For all $\Theta > 0$ there is $n_0 \in \mathbb{N}$ such that if $x, y \in X$ with $d_X(x, y) \leq \Theta$, then $\pi(x) = [x_j], \pi(y) = [x_{j'}]$ for some $j, j' \leq n_0$. In (other) words: If x and y belong to different orbits and their distance is $\leq \Theta$, then the orbits of x and y both are among the first $n_0 = n_0(\Theta)$ orbits.*

Proof. (cf. the proof of [AMN11, Theorem 1.1]) For all $A \subset X$, the *transporter* is defined as

$$G_{A,F} := \{g \in G \mid gA \cap F \neq \emptyset\}.$$

Since $G \curvearrowright X$ has finite point stabilisers, $G_{A,F}$ is finite if A is finite.

Step 1: Building a G -invariant (non-proper) metric d_1

The idea is to start with a (pseudo-)metric d_0 on X and to define d_1 by taking the sum of the values $d_0(gx, gy)$ over all $g \in G$ to enforce G -invariance. For this to work we must ensure that d_0 has finite support ‘along G -orbits’. Let

$$d_0(x, y) := \begin{cases} 0 & x = y \vee x, y \notin F \\ 1 & |\{x, y\} \cap F| = 1 \\ 2 & x \neq y \wedge x, y \in F. \end{cases}$$

Then d_0 is a pseudo-metric with values in $\mathbb{N}$. Next we define

$$d_1(x, y) := \sum_{g \in G} d_0(gx, gy).$$

If $g \notin G_{\{x, y\}, F}$, then, by definition, $\{gx, gy\} \cap F = \emptyset$ and $d_0(gx, gy) = 0$ follows. So the above sum is indeed finite. It is straightforward to check that $d_1 : X \times X \rightarrow \mathbb{N}$ is a G -invariant (but in general not a proper) metric on X .

Step 2: Building a G -invariant, orbitwise proper metric out of d_1

For our purposes a metric d on X is called *orbitwise proper* if for all $r > 0$ and $x \in X$ the image of $B_r^d(x)$, the (closed) ball of radius r around x , under $\pi : X \rightarrow G \backslash X$ is finite. Since X is countable, we can build a metric which is orbitwise proper by adding a summand to d_1 that forces orbits apart. Let $f : G \backslash X \rightarrow \mathbb{N}$ be defined as

$$f(\pi(x_i)) := \sum_{\nu=1}^i \nu = \frac{i^2 + i}{2}$$

and let

$$d_2(x, y) := d_1(x, y) + |f(\pi(x)) - f(\pi(y))|.$$

Then $d_2 : X \times X \rightarrow \mathbb{N}$ is a G -invariant metric as well. For $r > 0$ and $x \in X$, if $d_2(x, y) \leq r$ holds, in particular $|f(\pi(x)) - f(\pi(y))| \leq r$ has to hold. Hence, y can only lie in one of finitely many orbits and d_2 is orbitwise proper. Note that our special choice of f achieves even more: if $x, y \in X$ with $\pi(x_j) = \pi(x) \neq \pi(y) = \pi(x_{j'})$ and j or j' is $\geq \Theta + 1$, then $d_2(x, y) \geq \Theta + 1$.

Step 3: Building a G -invariant, proper generalised metric out of d_2

The metric d_3 is built by the measuring stick construction of [AMN11, Section 8]: Let $\mathcal{U} := \{gF \mid g \in G\}$. The graph $\Gamma_{\mathcal{U}}$ has $V(\Gamma_{\mathcal{U}}) := X$ as the set of vertices and has an edge (x, y) between x and y if $x, y \in U \in \mathcal{U}$, i.e. if x and y lie in the same translate of F . $\Gamma_{\mathcal{U}}$ is turned into a generalised metric space by assigning each edge (x, y) the length $d_2(x, y)$ and defining the distance between two points z and z' in $\Gamma_{\mathcal{U}}$ as the length of a shortest path in $\Gamma_{\mathcal{U}}$ between z and z' . By d_3 we denote the restriction of this generalised metric to X . It takes ∞ as a value if and only if $\Gamma_{\mathcal{U}}$ is not connected. A connected component of $\Gamma_{\mathcal{U}}$ will be called an *island*.

Since d_2 is G -invariant and G acts simplicially on $\Gamma_{\mathcal{U}}$, the generalised metric d_3 is G -invariant as well. By [AMN11, Lemma 8.7(b)] it is also proper. (The assumptions of this

lemma are satisfied since $d_2 =: d$ is a metric taking values in the integers, so in particular it is finite on islands and the Lebesgue number called for in the assumptions of [AMN11, Lemma 8.7(b)] can be taken to be $\frac{1}{2}$. Furthermore, d' appearing in the lemma is the result of the measuring stick construction applied to $d = d_2$ and $\mathcal{U}$ (cf. [AMN11, p. 629, before Proposition 8.3]). Therefore, it follows indeed that $d_3 = d'$ is proper.) Moreover, $d_3(x, y) \geq d_2(x, y)$ is satisfied for all $x, y \in X$.

Step 4: Building a G -invariant, proper metric out of d_3

By introducing bridges (of increasing length) between the islands (cf. [AMN11, Section 9]) one can obtain an actual metric: Since G is countable, there is a countable system of representatives $g_0 = 1, g_1, \dots$ of $G/\langle G_{F,F} \rangle$. A *bridge* in X is a 2-point set $\{gx, gg_ix\}$. Note that it is possible to have $\{gx, gg_ix\} = \{g'x, g'g_jx\}$ for $i \neq j$. The graph $\bar{\Gamma}_{\mathcal{U}}$ is obtained from $\Gamma_{\mathcal{U}}$ by adding for each bridge $\{gx, gg_ix\}$ one edge (x_1, x_2) to $\Gamma_{\mathcal{U}}$ connecting $gx, gg_ix \in X$. Such an edge (x_1, x_2) is assigned length

$$\max\{d_2(gx, gg_ix), i\},$$

where i is the smallest integer such that $\{gx, gg_ix\} = \{x_1, x_2\}$ holds for some $g \in G$. Furthermore, any edge coming from a bridge will also be called a bridge. The graph $\bar{\Gamma}_{\mathcal{U}}$ is connected. Let d_X be the path metric on $\bar{\Gamma}_{\mathcal{U}}$. In particular, d_X is still a G -invariant metric on X . Moreover, d_X is still proper: Since d_3 takes values in the integers and the length of a bridge is an integer as well, the metric d_X takes values in the integers. Hence, if $d_X(x, y) \leq R$, then a shortest path in $\bar{\Gamma}_{\mathcal{U}}$ between x and y can run through at most R edges. Obviously, $B_0^{d_X}(x)$ is finite. A point $y \in B_R^{d_X}(x)$ must be connected either by a bridge or by an edge contained in an island to a point $z \in B_{R-1}^{d_X}(x)$. Since d_3 is proper, there are only finitely many edges of length $< R$ that are contained in an island and start in a given $z \in B_{R-1}^{d_X}(x)$. Since G acts on X with finite point stabilisers, there are only finitely many bridges of length $< R$ starting from any given point. Thus, by induction $B_R^{d_X}(x)$ is finite.

Furthermore, for all $x, y \in X$ we have $d_X(x, y) \geq d_2(x, y)$: On islands of $\Gamma_{\mathcal{U}}$ we have $d_X(x, y) = d_3(x, y) \geq d_2(x, y)$, and thus the triangle inequality for d_2 forces the length of any path between x and y in $\bar{\Gamma}_{\mathcal{U}}$ to be at most $d_2(x, y)$. Hence, if there are $x, y \in X$ with $\pi(x) \neq \pi(y)$ and $d_X(x, y) \leq \Theta$, then the construction of d_2 forces x and y to lie in the first $n_0(\Theta) := \lceil \Theta + 1 \rceil$ orbits. $\square$

2.2 Setting up notation and the definition of Θ -small geodesics

Recall that we assume G to act k -acylindrically on T for some $k \geq 1$ and that we assume $V_0(T) = V(T) \setminus V_{\infty}(T)$, the set of vertices of finite degree, to be unbounded with respect to the path-metric d_T on T . Recall also that the term *geodesic* refers to both finite geodesics as well as (bi-infinite) geodesic rays in T .

We set up the following long list of carefully crafted choices and notations, some of which is illustrated in Figure 2.1.

$E^k(T)$	Denote by $E^k(T)$ the set of all geodesic segments of length k in T whose endpoints are vertices.
$E^k(T) \cap S$	For any subtree $S \subset T$ we write $\sigma \in E^k(T) \cap S$ if $\sigma \subset S$ and $\sigma \in E^k(T)$.
d_{E^k}	The set $E^k(T)$ is countable and the obvious action of G on $E^k(T)$ has finite point stabilisers, since the action $G \curvearrowright T$ is k -acylindrical. We fix a metric d_{E^k} on $E^k(T)$ according to Lemma 2.3.
σ lies on γ	We will often think of an element σ in $E^k(T)$ as an ‘elongated point’, which may or may not lie on a given geodesic γ . Thus, we will frequently use phrases like σ lies on γ , meaning that σ is a subset of the image of γ .
$o(\gamma), t(\gamma)$	For a finite geodesic $\gamma : [a, b] \rightarrow T$ we denote by $o(\gamma) := \gamma(a)$ and $t(\gamma) := \gamma(b)$ its start- and endpoint, respectively. We also extend this notation to (bi-infinite) geodesic rays in the canonical way.
$m(\gamma), m(\sigma)$	For a finite geodesic $\gamma : [a, b] \rightarrow T$ we denote by $m(\gamma) := \gamma(\frac{a+b}{2})$ its midpoint. Analogously, for an element $\sigma \in E^k(T)$ we denote by $m(\sigma)$ its midpoint. Note that, if k is even, the midpoint of σ is a vertex of T .
$\sigma_v(\gamma)$	For a geodesic γ and any vertex v on γ we denote (if it exists) by $\sigma_v(\gamma)$ the geodesic segment of length k on $[o(\gamma), v]$ that ends in v .
$v + i, v - i$	For a geodesic γ , any $i \in \mathbb{N}$ and a vertex v on γ we denote (if it exists) by $v + i$ (resp. $v - i$) the unique vertex on $[v, t(\gamma)]$ (resp. $[o(\gamma), v]$) that has distance i to v (with respect to d_T). If we want to specify along which geodesic we take this i th successor and i th predecessor, we write $v +_\gamma i$ and $v -_\gamma i$, respectively.
w_0, w'_0	We fix two vertices $w_0, w'_0 \in V_0(T)$ with $d_T(w_0, w'_0) \geq 5k$. (Such points exist since we assumed $V_0(T)$ to be unbounded with respect to d_T .)
θ_0	Let $\theta_0 := \max\{d_{E^k}(\sigma, \sigma') \mid \sigma, \sigma' \in E^k(T) \cap [w_0, w'_0]\}$.
m_ε	Choose your favourite $\varepsilon < \frac{1}{2}$. For the next two lines denote by m be the midpoint of $[w_0, w'_0]$. We denote (throughout the rest of this chapter) by m_ε the unique point on $[w_0, m]$ that has distance ε to m .
$a(g, \xi)$	Now for every $(g, \xi) \in G \times \Delta(T)$ only one of the two geodesics $[gw_0, \xi]$ and $[gw'_0, \xi]$ contains the point gm_ε . We denote by $a(g, \xi) \in \{gw_0, gw'_0\}$ the vertex such that $[a(g, \xi), \xi]$ contains gm_ε . Note that, for all $g, h \in G$, we have $ha(g, \xi) = a(hg, h\xi)$.

Figure 2.1: The vertices $v -_\gamma 2k$ and $v +_\gamma 1$ and the segment $\sigma_v(\gamma)$ on γ . The vertex $a(g, \xi)$ is well-defined, since the point m_ε is not a vertex.

Next we use the metric d_{E^k} to define when a geodesic is Θ -small.

Definition 2.4. Let $\Theta > 0$ and let γ be a geodesic in T . Any term of the form $d_{E^k}(\sigma, \sigma')$ for $\sigma, \sigma' \in E^k(T) \cap \text{im}(\gamma)$ with $\text{diam}_{d_T}(\sigma \cap \sigma') = k - 1$ is a *measurement (on γ)*. For a vertex v on γ the term $d_{E^k}(\sigma_v(\gamma), \sigma_{v+1}(\gamma))$ is called the *measurement at v (on γ)*. A measurement is Θ -small if it is $\leq \Theta$ and it is Θ -large if it is $> \Theta$. Finally, the geodesic γ is Θ -small if all measurements on γ are Θ -small.

Figure 2.2: $d_{E^k}(\sigma_v(\gamma), \sigma_{v+1}(\gamma))$ is the measurement at v on γ . Neither $d_{E^k}(\sigma_1, \sigma_2)$ nor $d_{E^k}(\sigma_3, \sigma_4)$ are measurements.

Note that any geodesic γ of length $\leq k$ is automatically Θ -small, since there are no measurements on γ .

Notation 2.5. For $\Theta > 0$ we define

$$G \times_{\Theta} \partial T := \{(g, \xi) \in G \times \partial T \mid [a(g, \xi), \xi] \text{ is } \Theta\text{-small}\}.$$

2.3 Constructing collections $\mathcal{W}_{\Theta}$ covering $G \times \Delta(T) \setminus G \times_{\Theta} \partial T$

This section provides for every $\Theta \geq \theta_0$ a G -invariant collection $\mathcal{W}_{\Theta}$ of open $\mathcal{F}_T$ -subsets of $G \times \Delta(T)$ of order 0 that covers $G \times \Delta(T) \setminus G \times_{\Theta} \partial T$. Given $\alpha > 0$, in general there is no single $\Theta = \Theta(\alpha)$ such that the collection $\mathcal{W}_{\Theta}$ is wide for $G \times \Delta(T) \setminus G \times_{\Theta} \partial T$. However, it is possible (cf. Section 2.4) to combine finitely many of these collections to obtain a new collection that is wide for some $G \times \Delta(T) \setminus G \times_{\Theta_{-1}} \partial T$ (with Θ_{-1} depending on α).

Definition 2.6. Let $\Theta \geq \theta_0$ and $v \in V(T)$. Define the set $W(v, \Theta) \subset G \times \Delta(T)$ to be

$$\begin{aligned} W(v, \Theta) := \{ & (g, \xi) \in G \times \Delta(T) \mid v \in [gm_{\varepsilon}, \xi], [a(g, \xi), v] \text{ is } \Theta\text{-small}, \\ & v = \xi \text{ or } d_{E^k}(\sigma_v(\gamma), \sigma_{v+\gamma 1}(\gamma)) > \Theta, \\ & \text{where } \gamma := [a(g, \xi), \xi] \}. \end{aligned}$$

This definition is illustrated below in Figure 2.3. Furthermore, note that, if $v \in [gm_{\varepsilon}, \xi]$, then the geodesic $[a(g, \xi), v]$ is of length $\geq 2k$ since we have chosen w_0, w'_0 with $d_T(w_0, w'_0) \geq 5k$. Hence, if $[v \in [gm_{\varepsilon}, \xi]$, then the segment $\sigma_v(\gamma)$ is always defined and, if additionally $v \neq \xi$, so is the segment $\sigma_{v+\gamma 1}(\gamma)$.

Lemma 2.7. *Let $\Theta \geq \theta_0$.*

- For all $v \in V(T)$, the set $W(v, \Theta) \subset G \times \Delta(T)$ is open.
- For all $g \in G$ and $v \in V(T)$, we have $gW(v, \Theta) = W(gv, \Theta)$.
- If $v \neq w \in V(T)$, then $W(v, \Theta) \cap W(w, \Theta) = \emptyset$.

Proof. a) Let $(g, \xi) \in W(v, \Theta)$ and let $\gamma := [a(g, \xi), \xi]$. First, let $v \neq \xi$, and thus $[a(g, \xi), v +_\gamma 1] \subset [a(g, \xi), \xi]$. Note that in this case any point (g, ξ') with

$$[a(g, \xi), v +_\gamma 1] = [a(g, \xi'), v +_\gamma 1] \subseteq [a(g, \xi'), \xi']$$

lies in $W(v, \Theta)$ as well (since the measurements relevant for $(g, \xi) \in W(v, \Theta)$ are the same measurements relevant for $(g, \xi') \in W(v, \Theta)$). Set $U := M(\xi, v) \cap \Delta(T)$. Then, for all $\xi' \in U$, the geodesic $[a(g, \xi), \xi']$ contains $v +_{\gamma} 1$, and hence gm_{ε} as well. Therefore, $[a(g, \xi), v +_{\gamma} 1] = [a(g, \xi'), v +_{\gamma} 1]$ and $(g, \xi) \in W(v, \Theta)$ follows.

If $v = \xi$, then we construct U as follows: Let $\sigma := \sigma_v([a(g, \xi), v])$. Since d_{E^k} is proper, there are only finitely many σ' with $d_{E^k}(\sigma, \sigma') \leq \Theta$ and $\text{diam}_{d_T}(\sigma \cap \sigma') = k - 1$. Denote them by $\sigma'_1, \dots, \sigma'_m$. Set

$$U := M(v, \{t(\sigma'_1), \dots, t(\sigma'_m)\}) \cap \Delta(T).$$

Since $[a(g, \xi), v]$ is Θ -small by assumption, the segment $\sigma_{v-\gamma 1}(\gamma)$ is some σ'_i , and hence $v-\gamma 1$ is $t(\sigma'_i)$. Therefore, for any $\xi' \in U$, the geodesic $[a(g, \xi), \xi']$ contains $[a(g, \xi), v]$ and, in particular, gm_ε . It follows that $[a(g, \xi'), v] = [a(g, \xi), v]$, which is Θ -small by assumption. And our choice of the σ'_i guarantees that the measurement at v on $[a(g, \xi'), \xi']$ is Θ -large.

- b) This follows from the G -invariance of d_{E^k} .
- c) Assume we have $v \neq w$ and $(g, \xi) \in W(v, \Theta) \cap W(w, \Theta)$. Then, by definition of $W(v, \Theta)$ and $W(w, \Theta)$, both v and w lie on $[gm_{\varepsilon}, \xi]$. Hence, without loss of generality we may assume v lies on $[gm_{\varepsilon}, w]$. Now, on the one hand, $(g, \xi) \in W(w, \Theta)$ implies that the measurement at v on $[a(g, \xi), w]$ is Θ -small. On the other hand, $(g, \xi) \in W(v, \Theta)$ implies that the measurement at v on $[a(g, \xi), \xi]$ is Θ -large. $\square$

Figure 2.3: The point (g, ξ) lies in $W(v, \Theta)$.

For future reference we state the following lemma.

Lemma 2.8. *Let $\Theta \geq \theta_0$. Then $\mathcal{W}_\Theta := \{W(v, \Theta) \mid v \in V(T)\}$ is a G -invariant collection of open $\mathcal{F}_T$ -subsets of $G \times \Delta(T)$ and covers $G \times \Delta(T) \setminus G \times_\Theta \partial T$. Furthermore, the order of $\mathcal{W}_\Theta$ is 0.*

Proof. By Lemma 2.7 it is clear that $\mathcal{W}_\Theta$ is an open, G -invariant collection of order 0. Moreover, by combining Lemma 2.7 b) and c), it follows that each $W(v, \Theta)$ is an $\mathcal{F}_T$ -subset of $G \times \Delta(T)$. The only property left to show is that the collection $\mathcal{W}_\Theta$ covers the set $G \times \Delta(T) \setminus G \times_\Theta \partial T$: Let $(g, \xi) \in G \times \Delta(T) \setminus G \times_\Theta \partial T$ and $\gamma := [a(g, \xi), \xi]$. If γ is Θ -small, then $\xi \in V(T)$, and hence $(g, \xi) \in W(\xi, \Theta)$. If γ is not Θ -small, then there is a unique vertex v on γ such that $[a(g, \xi), v]$ is Θ -small, but $[a(g, \xi), v +_\gamma 1]$ is not. Note that v might lie on $[gw_0, gw'_0]$, but due to our choice of θ_0 the vertex v can not lie between $a(g, \xi)$ and gm_ε . Thus, $v \in [gm_\varepsilon, \xi]$ and $(g, \xi) \in W(v, \Theta)$ holds. $\square$

2.4 A collection that is wide for $G \times \Delta(T) \setminus G \times_{\Theta_{-1}} \partial T$

The aim of this section is to show that—given $\alpha > 0$ —we can combine $k + 2$ collections of the form $\mathcal{W}_{\Theta_i}$ in order to obtain a collection of order $\leq k + 1$ that is wide for a suitable $G \times \Delta(T) \setminus G \times_{\Theta_{-1}} \partial T$, where the value Θ_{-1} depends on the given $\alpha > 0$.

Proposition 2.9. *There is $N \in \mathbb{N}$ such that for all $\alpha > 0$ there are $\Theta_i = \Theta_i(\alpha)$ for $-1 \leq i \leq N$ such that for all $(1, \xi) \in G \times \Delta(T)$ one of the following two statements holds:*

- a) *there is $v \in V(T)$ such that $B_\alpha(1) \times \{\xi\} \subset W(v, \Theta_i)$ for some $0 \leq i \leq N$;*
- b) *$\xi \in \partial T$ and the geodesic $[a(1, \xi), \xi]$ is Θ_{-1} -small.*

Before entering the rather technical proof, we outline its underlying ideas:

For any $\Theta \geq \theta_0$ we already know that for each point $(1, \xi) \in G \times \Delta(T) \setminus G \times_\Theta \partial T$ we can find a vertex $v \in V(T)$ by Lemma 2.8 such that $(1, \xi) \in W(v, \Theta)$ holds. We do not expect $[a(h, \xi), v]$ to be Θ -small for $h \in B_\alpha(1)$ as well, but we wish to calculate how small the geodesics $[a(h, \xi), v]$ are, given that $[a(1, \xi), v]$ is Θ -small.

In the case $k = 1$ this results in the following type of picture:

Since $B_\alpha(1)$ is finite, the tree L_0 spanned by $\{a(h, \xi) \mid h \in B_\alpha(1)\}$ is as well. Thus, there is some Υ independent of h such that $d_{E^1}(\sigma_0, \sigma_1) \leq \Upsilon$. Since $d_{E^1}(\sigma_0, \sigma_2) \leq \Theta$ by virtue of $(1, \xi) \in W(v, \Theta)$, we can estimate $d_{E^1}(\sigma_1, \sigma_2)$ to be $\leq \Theta + \Upsilon$. Hence, for all $h \in B_\alpha(1)$

the geodesic $[a(h, \xi), v]$ is $\Theta + \Upsilon$ -small. If the measurement at v on $[a(1, \xi), \xi]$ is not only Θ -large, but in fact $\Theta + \Upsilon$ -large, we can conclude $B_\alpha(1) \times \{\xi\} \subset W(v, \Theta + \Upsilon)$. However, in general, the measurement at v on $[a(1, \xi), \xi]$ is not $\Theta + \Upsilon$ -large and this requires that we ‘push v further to the right’. More precisely, we search for the first vertex w on $[a(1, \xi), \xi]$ such that $[a(1, \xi), w]$ is $\Theta + \Upsilon$ -small, but the measurement at w is not. If such w exists, then $B_\alpha(1) \times \{\xi\} \subset W(w, \Theta + \Upsilon)$ follows. If such w does not exist, then $[a(1, \xi), \xi]$ is $\Theta + \Upsilon$ -small. Thus, if $\xi \in \partial T$, we are in case b) of the proposition. Finally, if $\xi \in V_\infty(T)$, our tailored definition of the sets $W(v, \Theta + \Upsilon)$ implies that $B_\alpha(1) \times \{\xi\} \times W(\xi, \Theta + \Upsilon)$ holds. Since we only used sets of the form $W(-, \Theta + \Upsilon)$ to ensure that either a) or b) holds for all $(1, \xi) \in G \times \Delta(T)$, this basically concludes the proof of the above proposition for the case $k = 1$.

In the case $k \geq 2$, however, testing if the geodesics $[a(h, \xi), v]$ are Θ -small is more involved since we will have to make statements about pictures of the following type:

We still know that L_0 (as defined before) is finite. Thus, we still have a bound Υ on all measurements on $[a(1, \xi), a(h, \xi)]$. However, we do not have a triangle inequality anymore to give an estimate for $d_{E^k}(\sigma_1, \sigma_2)$ based on Υ and the fact that $d_{E^k}(\sigma_0, \sigma_3) \leq \Theta$. To still obtain some estimate we use that d_{E^k} is proper: since $[a(1, \xi), v]$ is Θ -small, has a segment of length $\geq k$ in common with L_0 and the latter is finite, there are only finitely many possibilities for σ_3 . Adding all possible σ_3 to L_0 still results in a finite tree L_1 and we find some bound Υ' on the measurements lying on L_1 . This obviates the need for a triangle inequality, since now both σ_1 and σ_2 lie on L_1 . Hence, $d_{E^k}(\sigma_1, \sigma_2)$ is Υ' -small. The idea of ‘pushing v further to the right if necessary’ stays the same, but various possible constellations of how the resulting vertex w lies relative to v necessitates more than one iteration step and careful case distinctions.

Next, we make precise how to define an enlargement of L_0 suitable for all iteration steps.

Choosing Constants 2.10. Let $\alpha > 0$. The following notions all depend on α . However, since we will never compare different values of α , this dependence is not reflected in the notation.

L_0	The set $B_\alpha(1)$ is finite. As is the set $M := \{hw_0, hw'_0 \mid h \in B_\alpha(1)\}$. Let $L_0 \subset T$ be the subtree of T spanned by M . In particular, L_0 is a finite subtree of T .
Υ_0	We define Υ_0 as the diameter of the set $E^k(T) \cap L_0$ with respect to the metric d_{E^k} . Thus, for all $\sigma, \sigma' \in E^k(T) \cap L_0$ the inequality $d_{E^k}(\sigma, \sigma') \leq \Upsilon_0$ holds. Note that Υ_0 automatically has to be $\geq \theta_0$ as $[w_0, w'_0] \subset L_0$.

$L_k(\chi)$	For $\chi > 0$ we construct the constant $\Upsilon_1(\chi)$ as follows: We enlarge $L_0 =: L_0(\chi)$ iteratively: For $1 \leq \nu \leq k$ let $L_\nu(\chi)$ be the union of $L_{\nu-1}(\chi)$ and all $\sigma \in E^k(T)$ such that there is $\sigma' \in E^k(T) \cap L_{\nu-1}(\chi)$ with $\sigma \cap \sigma' \neq \emptyset$ and $d_{E^k}(\sigma, \sigma') \leq \chi$. Since, to obtain $L_\nu(\chi)$ from $L_{\nu-1}(\chi)$, we only add segments that have non-trivial intersection with $L_{\nu-1}(\chi)$, each $L_\nu(\chi)$ is again a subtree of T . Since L_0 is finite and d_{E^k} is proper, the tree $L_\nu(\chi)$ is finite for all $1 \leq \nu \leq k$.
$\Upsilon_1(\chi)$	We define $\Upsilon_1(\chi)$ as the diameter of the set $E^k(T) \cap L_k(\chi)$ with respect to the metric d_{E^k} . Thus, for all $\sigma, \sigma' \in E^k(T) \cap L_k(\chi)$ the inequality $d_{E^k}(\sigma, \sigma') \leq \Upsilon_1(\chi)$ holds. Note that $\Upsilon_1(\chi) \geq \Upsilon_0$.
$\Upsilon_m(\chi)$	Let $\chi > 0$. For $m \geq 0$ we define $\Upsilon_m(\chi)$ iteratively by $\Upsilon_m(\chi) := \Upsilon_1(\Upsilon_{m-1}(\chi))$.

The proof of Proposition 2.9 requires some more notation.

Notation 2.11. For a given $\alpha > 0$ and $(1, \xi) \in G \times \Delta(T)$ we fix the following notation that depends on α and ξ (cf. Figure 2.4 for an illustration of some of the notation). The naming will not always reflect both of these dependencies, though.

γ_h	For $h \in B_\alpha(1)$ let $\gamma_h := [a(h, \xi), \xi]$.
v_h	For $h \neq 1$ let v_h be the unique first vertex that lies on both γ_1 and γ_h . Hence, we have $\gamma_1 \cap \gamma_h = [v_h, \xi]$. Note that the case $v_h = \xi$ is possible.
B_v^{inn}	For any vertex v of γ_1 we set $B_v^{\text{inn}} := \{h \in B_\alpha(1) \mid v \in V(\gamma_h)\}$. Thus, $h \in B_\alpha(1)$ is an element of B_v^{inn} if and only if γ_h contains v , which is the case if and only if $[v, \xi] \subseteq [v_h, \xi]$.
B_v^{ext}	Furthermore, let $B_v^{\text{ext}} := B_\alpha(1) \setminus B_v^{\text{inn}}$. Thus, $h \in B_\alpha(1)$ is an element of B_v^{ext} if and only if $v \notin [v_h, \xi]$.
$v_0(\chi)$	For any $\chi > \theta_0$ denote—if it exists—by $v_0(\chi)$ the first vertex on γ_1 such that $[a(1, \xi), v_0(\chi)]$ is χ -small, but $[a(1, \xi), v_0(\chi) +_{\gamma_1} 1]$ is not. In particular, the existence of $v_0(\chi) +_{\gamma_1} 1$ is necessary for the existence of $v_0(\chi)$. Note that $d_T(a(1, \xi), v_0(\chi)) \geq 2k$: Since $[w_0, w'_0]$ is θ_0 -small, it follows that $m_\varepsilon \in [a(1, \xi), v_0(\chi)]$ and $[a(1, \xi), m_\varepsilon]$ has length $\geq 2k$.

Figure 2.4: $h_1, h_2, h_3 \in B_\alpha(1)$ belong to B_v^{inn} , but $h_4 \in B_\alpha(1)$ belongs to B_v^{ext} .

We proceed by showing a series of technical lemmata, whose sole purpose is to increase readability of the proof of Proposition 2.9. For all statements we will use extensively the constants defined in Choosing Constants 2.10 and assume that α and $(1, \xi)$ are fixed. While looking at the pictures in the proofs it is also important to keep in mind the following observation: Our definition of $W(v, \Theta)$ allows for ξ to lie on $[w_0, w'_0]$. However, in what follows we often assume the existence of a $v_0(\chi)$ for some $\chi \geq \Upsilon_0$, which forces ξ to lie outside of $[w_0, w'_0]$.

Lemma 2.12. *Let $\chi \geq \Upsilon_0$. If $v_0(\chi)$ exists, then $B_{v_0(\chi)}^{\text{inn}} = B_\alpha(1)$.*

Proof. Assume there is $h \in B_\alpha(1)$ such that $h \notin B_{v_0(\chi)}^{\text{inn}}$. Thus, $v_h \in (v_0(\chi), \xi]$. Then by uniqueness of geodesics in T and the construction of Υ_0 it follows that $[a(1, \xi), v_0(\chi) +_{\gamma_1} 1]$ is Υ_0 -small (cf. Figure 2.4 with $h = h_4$ and $v_0(\chi) = v$). A contradiction. $\square$

In general, the ‘concatenation’ of two χ -small geodesics $[x, y]$ (ending in a vertex of T) and $[y, z]$ (starting in a vertex of T) is not χ -small. However, if two χ -small geodesics overlap in a sufficiently large terminal segment, we can conclude in this way. This is made precise in the following lemma, whose proof is entirely encapsulated in Figure 2.5.

Lemma 2.13. *Let $\chi \geq 0$ and let $[x, z]$ be a geodesic. If there are y, y' on $[x, z]$ such that the intersection $[x, y] \cap [y', z] = [y', y]$ has length $\geq k$ and both $[x, y]$ and $[y', z]$ are χ -small, then $[x, z]$ is χ -small.*

Figure 2.5: The ‘union’ of two χ -small geodesic overlapping in a segment $\geq k$ is again χ -small.

In the sequel we will use this way of reasoning without explicitly referencing the above lemma.

Lemma 2.14. *Let $\chi \geq \Upsilon_0$ and x be a vertex on γ_1 such that $[a(1, \xi), x]$ is χ -small. Then the geodesic $[a(h, \xi), x]$ is $\Upsilon_1(\chi)$ -small for all $h \in B_\alpha(1)$.*

Proof. If $h \in B_x^{\text{ext}}$, then $[a(h, \xi), x] \subset [a(h, \xi), a(1, \xi)] \subset L_0$. Hence, $[a(h, \xi), x]$ is Υ_0 -small. If $h \in B_x^{\text{inn}}$, then we distinguish cases upon where v_h lies on $[a(1, \xi), x]$:

a) This case is clear:

Thus, we assume from now on, that $a(h, \xi) \notin [a(1, \xi), x]$.

b) This one is clear as well:

c) If v_h lies in the interior of $[a(1, \xi), x]$ and $[a(1, \xi), v_h]$ has length $\geq k$,

then the first k edges on $[v_h, x]$ (if there are this many) are added to L_0 in the process of building $L_k(\chi)$, since $[a(1, \xi), v_h]$ is χ -small (cf. Choosing Constants 2.10). Thus, $[a(h, \xi), v_h +_{\gamma_1} k]$ (resp. $[a(h, \xi), x]$ if $[v_h, x]$ is of length $< k$) is $\Upsilon_1(\chi)$ -small. Since $[v_h, x]$ was χ -small by assumption, it follows that $[a(h, \xi), x]$ is $\Upsilon_1(\chi)$ -small.

d) If v_h lies in the interior of $[a(1, \xi), x]$ and $[a(1, \xi), v_h]$ has length $< k$, there are two subcases.

1) The subcase where $x \in [a(1, \xi), m_\varepsilon]$ is trivial:

2) The subcase where $x \notin [a(1, \xi), m_\varepsilon]$ looks as follows:

Since $[v_h, m_\varepsilon]$ has length $\geq k$, $[a(h, \xi), m_\varepsilon]$ is Υ_0 -small and $[v_h, x]$ is χ -small, the geodesic $[a(h, \xi), x]$ is χ -small.

e) This case is trivial again:

f) And finally:

Again, the geodesic $[a(h, \xi), x]$ is χ -small since $[a(h, \xi), m_\varepsilon]$ is Υ_0 -small, $[a(1, \xi), x]$ is χ -small and $[a(1, \xi), m_\varepsilon]$ has length $\geq k$. $\square$

Lemma 2.15. *Let $\chi \geq \Upsilon_0$. If $v_0(\chi)$ exists and $B_{v_0(\chi) - \gamma_1 k}^{\text{inn}} = B_\alpha(1)$, then one of the following statements holds:*

- 1) $B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$;
- 2) $v_0(\Upsilon_1(\chi))$ does not exist, i.e. $[a(1, \xi), \xi]$ is $\Upsilon_1(\chi)$ -small;
- 3) $v_0(\Upsilon_1(\chi))$ exists and $B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_1(\chi)), \Upsilon_1(\chi))$.

Proof. In this proof we abbreviate d_{E^k} to d . For all $h \in B_\alpha(1)$, the geodesic $[a(h, \xi), v_0(\chi)]$ is $\Upsilon_1(\chi)$ -small by Lemma 2.14 (applied to $x = v_0(\chi)$). Since $B_\alpha(1) = B_{v_0(\chi) - \gamma_1 k}^{\text{inn}}$, the geodesics $[v_h, v_0(\chi)] \subset \gamma_h$ all have length $\geq k$. Hence, for all $h \in B_\alpha(1)$, the measurement at $v_0(\chi)$ on γ_1 is the measurement at $v_0(\chi)$ on γ_h . Thus, if the measurement at $v_0(\chi)$ on γ_1 is $\Upsilon_1(\chi)$ -large, so is the measurement at $v_0(\chi)$ on γ_h and $B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$ follows, which is statement 1) above.

Assume from now on that the measurement at $v_0(\chi)$ on γ_1 is $\Upsilon_1(\chi)$ -small. If $v_0(\Upsilon_1(\chi))$ does not exist, then $[a(1, \xi), \xi]$ is $\Upsilon_1(\chi)$ -small, which is statement 2) above. If $v_0(\Upsilon_1(\chi))$ exists, it lies in $(v_0(\chi), \xi]$. Thus, the intersection of the $\Upsilon_1(\chi)$ -small geodesic $[a(h, \xi), v_0(\chi)]$ with the $\Upsilon_1(\chi)$ -small geodesic $[v_0(\chi) - \gamma_1 k, v_0(\Upsilon_1(\chi))]$ has length $\geq k$ for all $h \in B_\alpha(1)$. Thus, all $[a(h, \xi), v_0(\Upsilon_1(\chi))]$ are still $\Upsilon_1(\chi)$ -small and the measurement at $v_0(\Upsilon_1(\chi))$ on γ_1 is the measurement at $v_0(\Upsilon_1(\chi))$ on γ_h . Hence, $B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_1(\chi)), \Upsilon_1(\chi))$, which is statement 3) above. $\square$

Lemma 2.16. *Let $\chi \geq \Upsilon_0$. Assume that $v_0(\chi)$ exists and let $0 \leq r \leq k-1$ such that $B_{v_0(\chi)-\gamma_1 r}^{\text{ext}} = \emptyset$, but $B_{v_0(\chi)-\gamma_1(r+1)}^{\text{ext}} \neq \emptyset$. Then one of the following statements holds:*

- 1.) $B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$;
- 2.) $v_0(\Upsilon_2(\chi))$ does not exist, i.e. $[a(1, \xi), \xi]$ is $\Upsilon_2(\chi)$ -small;
- 3.) $v_0(\Upsilon_2(\chi))$ exists and there is an $s \geq 1$ such that $B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s)}^{\text{ext}} = \emptyset$, but $B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s+1)}^{\text{ext}} \neq \emptyset$.

Proof. In this proof we abbreviate d_{E^k} to d again. As in the previous lemma, for all $h \in B_\alpha(1)$ the geodesics $[a(h, \xi), v_0(\chi)]$ is $\Upsilon_1(\chi)$ -small by Lemma 2.14 (applied to $x = v_0(\chi)$). The measurement at $v_0(\chi)$ on $[a(h, \xi), \xi]$ is

$$d(\sigma_{v_0(\chi)}(\gamma_h), \sigma_{v_0(\chi)+1}(\gamma_h)).$$

Note that, since $B_{v_0(\chi)}^{\text{inn}} = B_\alpha(1)$ by Lemma 2.12, $v_0(\chi) + \gamma_1 1 = v_0(\chi) + \gamma_h 1$. So the $+1$ in the second argument above is unambiguous. However, in general $\sigma_{v_0(\chi)+1}(\gamma_h) \neq \sigma_{v_0(\chi)+1}(\gamma_1)$ and this is the point where we have to put in a little work.

If for all $h \in B_\alpha(1)$ the measurement at $v_0(\chi)$ on γ_h is $\Upsilon_1(\chi)$ -large, then

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$$

by definition of $W(v_0(\chi), \Upsilon_1(\chi))$, and this is statement 1) above. So assume from now on that there is an $h_1 \in B_\alpha(1)$ such that the measurement at $v_0(\chi)$ on γ_{h_1} is $\Upsilon_1(\chi)$ -small. Moreover, we can assume that $v_0(\Upsilon_2(\chi))$ does exist, since the case that $v_0(\Upsilon_2(\chi))$ does not exist, is exactly statement 2) above.

Since $B_{v_0(\chi)-\gamma_1 r}^{\text{ext}} = \emptyset$, but $B_{v_0(\chi)-\gamma_1(r+1)}^{\text{ext}} \neq \emptyset$, there is $h_0 \in B_\alpha(1)$ with $v_{h_0} = v_0(\chi) - \gamma_1 r$. So for this h_0 the geodesic $[a(1, \xi), a(h_0, \xi)]$ runs through $v_0(\chi) - \gamma_1 r$. Since, for all $h \in B_\alpha(1)$, the vertex v_h is contained in $[a(1, \xi), v_{h_0}]$, it follows that $[a(h, \xi), v_0(\chi) - \gamma_1 r] \subset L_0$ holds for all h . Assume $h_1 m_\varepsilon \notin [a(h_1, \xi), v_0(\chi)]$. Then we must have $[a(1, \xi), v_0(\chi) + 1] \subset L_0$ - a contradiction to the definition of $v_0(\chi)$, since $\chi \geq \Upsilon_0$. Thus, $h_1 m_\varepsilon \in [a(h_1, \xi), v_0(\chi)]$, and hence $d_T(a(h_1, \xi), v_0(\chi) - \gamma_1 r) \geq k$. Combining the facts that $[a(h_1, \xi), v_0(\chi)]$ is $\Upsilon_1(\chi)$ -small (by Lemma 2.14) and that $r+1 \leq k$ with our assumption that the measurement at $v_0(\chi)$ on γ_{h_1} is $\Upsilon_1(\chi)$ -small, it follows that the segment $[v_0(\chi) - \gamma_1 r, v_0(\chi) + \gamma_1 1]$ is contained in $L_k(\Upsilon_1(\chi))$ (cf. Figure 2.6 and Choosing Constants 2.10). Note that, if $r = k-1$, then we need all k steps in the iteration process used to build $L_k(\Upsilon_1(\chi))$ from L_0 in order to add this whole segment. Recall that $\Upsilon_2(\chi) = \Upsilon_1(\Upsilon_1(\chi))$. Hence, $[a(h, \xi), v_0(\chi) + 1] \subset L_k(\Upsilon_1(\chi))$ is $\Upsilon_2(\chi)$ -small for all $h \in B_\alpha(1)$. In particular, the vertex $v_0(\Upsilon_2(\chi))$ has to lie on $(v_0(\chi), \xi]$ and $s := d_T(v_0(\chi), v_0(\Upsilon_2(\chi)))$ is ≥ 1 . Moreover, it is

$$\begin{aligned} B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s)}^{\text{ext}} &= B_{v_0(\chi)-\gamma_1 r}^{\text{ext}} = \emptyset \quad \text{and} \\ B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s+1)}^{\text{ext}} &= B_{v_0(\chi)-\gamma_1(r+1)}^{\text{ext}} \neq \emptyset. \end{aligned}$$

□

Figure 2.6: The segment $[v_0 - \gamma_1 r, v_0 + \gamma_1 1]$, with $v_0 = v_0(\chi)$, is contained in $L_k(\Upsilon_1(\chi))$.

We are now ready to prove Proposition 2.9.

Proposition 2.9. *There is $N \in \mathbb{N}$ such that for all $\alpha > 0$ there are $\Theta_i = \Theta_i(\alpha)$ for $-1 \leq i \leq N$ such that for all $(1, \xi) \in G \times \Delta(T)$ one of the following two statements holds:*

- a) *there is $v \in V(T)$ such that $B_\alpha(1) \times \{\xi\} \subset W(v, \Theta_i)$ for some $0 \leq i \leq N$;*
- b) *$\xi \in \partial T$ and the geodesic $[a(1, \xi), \xi]$ is Θ_{-1} -small.*

Proof. Set $N := k + 1$ and let $\alpha > 0$ be given. Further define the following constants depending on α :

- Let Υ_0 be as in Choosing Constants 2.10.
- For $0 \leq i \leq k$ define $\Theta_i := \Upsilon_{2i+1}(\Upsilon_0)$ (also cf. Choosing Constants 2.10).
- Choose Θ_{-1} such that $\Theta_{-1} > \Upsilon_{2(k+1)}(\Upsilon_0)$. Note that the iterative construction of $\Upsilon_{2(k+1)}(\Upsilon_0)$ guarantees $\Theta_{-1} \geq \Upsilon_i(\Upsilon_0)$ for $1 > i \leq 2k + 1$ and, therefore, also $\Theta_{-1} > \Upsilon_0$.
- Define $\Theta_{k+1} := \Upsilon_1(\Theta_{-1})$.

Let $(1, \xi) \in G \times \Delta(T)$ and recall the notation $\gamma_h, B_v^{\text{inn}}$ and B_v^{ext} from Notation 2.11.

Case 1: Assume that γ_1 is Θ_{-1} -small. If $\xi \in \partial T$, then statement b) above holds. If $\xi \in V_\infty(T)$, then by Lemma 2.14 (for $x = \xi$ and $\chi = \Theta_{-1}$) and the definition of $W(\xi, \Upsilon_1(\Theta_{-1}))$ it follows that

$$B_\alpha(1) \times \{\xi\} \subset W(\xi, \Upsilon_1(\Theta_{-1})) = W(\xi, \Theta_{k+1}).$$

Case 2: Assume that γ_1 is not Θ_{-1} -small. In particular, γ_1 is not Υ_0 -small and $v_0(\Upsilon_0)$ exists. Thus, by Lemma 2.12, $B_{v_0(\Upsilon_0)}^{\text{ext}} = \emptyset$ and there is a unique $r_0 \geq 0$ such that

$$B_{v_0(\Upsilon_0) - \gamma_1 r_0}^{\text{ext}} = \emptyset, \text{ but } B_{v_0(\Upsilon_0) - \gamma_1(r_0+1)}^{\text{ext}} \neq \emptyset.$$

If $r_0 \geq k$, we can apply Lemma 2.15 with $\chi = \Upsilon_0$. Since γ_1 is not $\Upsilon_1(\Upsilon_0)$ -small, it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_0), \Upsilon_1(\Upsilon_0)) = W(v_0(\Upsilon_0), \Theta_0)$$

or that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_1(\Upsilon_0)), \Upsilon_1(\Upsilon_0)) = W(v_0(\Upsilon_1(\Upsilon_0)), \Theta_0).$$

If $r_0 \leq k - 1$, we can apply Lemma 2.16 with $r = r_0$ and $\chi = \Upsilon_0$. Since γ_1 is not $\Upsilon_2(\Upsilon_0)$ -small, it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_0), \Upsilon_1(\Upsilon_0)) = W(v_0(\Upsilon_0), \Theta_0)$$

or that $v_0(\Upsilon_2(\Upsilon_0))$ exists and that there is an $s_0 \geq 1$ such that

$$B_{v_0(\Upsilon_2(\Upsilon_0))-\gamma_1(r_0+s_0)}^{\text{ext}} = \emptyset, \text{ but } B_{v_0(\Upsilon_2(\Upsilon_0))-\gamma_1(r_0+s_0+1)}^{\text{ext}} \neq \emptyset.$$

This process can be iterated for $0 \leq i \leq k$. Assume that we have already constructed an $r_i \geq 0$ with

$$B_{v_0(\Upsilon_{2i}(\Upsilon_0))-\gamma_1 r_i}^{\text{ext}} = \emptyset, \text{ but } B_{v_0(\Upsilon_{2i}(\Upsilon_0))-\gamma_1(r_i+1)}^{\text{ext}} \neq \emptyset,$$

where for $i = 0$ we interpret $\Upsilon_0(\Upsilon_0)$ as Υ_0 . If $r_i \geq k$, we can apply Lemma 2.15 with $\chi = \Upsilon_{2i}(\Upsilon_0)$. Since γ_1 is not $\Upsilon_{2i+1}(\Upsilon_0)$ -small (as long as $i \leq k$), it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_{2i}(\Upsilon_0)), \Upsilon_{2i+1}(\Upsilon_0))$$

or that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_{2i+1}(\Upsilon_0)), \Upsilon_{2i+1}(\Upsilon_0)).$$

If $r_i \leq k - 1$, we can apply Lemma 2.16 with $r = r_i$ and $\chi = \Upsilon_{2i}(\Upsilon_0)$. Since γ_1 is not $\Upsilon_{2(i+1)}(\Upsilon_0)$ -small (as long as $i \leq k$), it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_{2i}(\Upsilon_0)), \Upsilon_{2i+1}(\Upsilon_0))$$

or that $v_0(\Upsilon_{2(i+1)}(\Upsilon_0))$ exists and that there is $s_i \geq 1$ such that

$$B_{v_0(\Upsilon_{2(i+1)}(\Upsilon_0))-\gamma_1(r_i+s_i)}^{\text{ext}} = \emptyset, \text{ but } B_{v_0(\Upsilon_{2(i+1)}(\Upsilon_0))-\gamma_1(r_i+s_i+1)}^{\text{ext}} \neq \emptyset.$$

In this case we define $r_{i+1} := r_i + s_i$ and repeat the process. Since in each step $s_i \geq 1$, there is $0 \leq j \leq k$ such that $r_j \geq k$. Hence, all constants $\Upsilon_{2i+1}(\Upsilon_0)$ considered in the process are among $\Theta_0, \dots, \Theta_k$. $\square$

This concludes the construction of suitable G -invariant collections of open $\mathcal{F}$ -subsets that are wide for $G \times \Delta(T) \setminus G \times_{\Theta_{-1}} \partial T$ (with Θ_{-1} depending on the given $\alpha > 0$).

2.5 Another proper G -invariant metric on $E^k(T)$

The aim of this and the following two sections is to construct—for all Θ —a G -invariant collection of open $\mathcal{F}$ -subsets that is wide for $G \times_\Theta \partial T$. We will do this analogously to [Bar16]. The first step to this, provided in this section, is to define another proper G -invariant metric on $E^k(T)$ which essentially coincides with the path-metric d_T when restricted to segments lying on a small geodesic.

From now on assume that k is even and recall that in this case the midpoint $m(\sigma)$ of any $\sigma \in E^k(T)$ is a vertex of T .

Definition 2.17. Let $\Theta > 0$ and let $\sigma, \sigma' \in E^k(T)$. The geodesic $\gamma = [m(\sigma), m(\sigma')]$ is called *extended Θ -small*, if the following holds:

- γ is Θ -small (as in Definition 2.4).
- If $d_T(m(\sigma), m(\sigma')) \geq k$, then

$$d_{E^k}(\sigma, \sigma_{o(\gamma)+\gamma k}) \leq \Theta$$

and

$$d_{E^k}(\sigma_{t(\gamma)}, \sigma') \leq \Theta.$$

- If $d_T(m(\sigma), m(\sigma')) \leq k - 1$, then $d_{E^k}(\sigma, \sigma') \leq \Theta$.

Remember that the $+\gamma$ -operation on vertices of γ is meant as introduced in Section 2.2.

Definition 2.18. An *oriented segment (of length k)* in T is an element $\sigma \in E^k(T)$ together with a vertex $b \in \{o(\sigma), t(\sigma)\}$. For $\sigma, \sigma' \in E^k(T)$ denote by $\mathcal{M}_\Theta(\sigma, \sigma')$ the set of all sequences $(\sigma_0, b_0), (\sigma_1, b_1), \dots, (\sigma_n, b_n)$ of oriented segments of length k in T such that $\sigma = \sigma_0$ and $\sigma' = \sigma_n$ (as unoriented segments) and the geodesics $[m(\sigma_i), m(\sigma_{i+1})]$ are extended Θ -small. For brevity, elements of $\mathcal{M}_\Theta(\sigma, \sigma')$ are called *admissible sequences (for the pair (σ, σ'))* and the vertices b_i are dropped from the notation. Finally, we define

$$d_\Theta(\sigma, \sigma') := \min_{\mathcal{M}_\Theta(\sigma, \sigma')} \sum_{i=0}^{n-1} (d_T(m(\sigma_i), m(\sigma_{i+1})) + d_T(o(\sigma_i), o(\sigma_{i+1})) + d_T(t(\sigma_i), t(\sigma_{i+1}))),$$

where d_T is the path-metric on T .

Lemma 2.19. d_Θ is a proper G -invariant generalised metric on $E^k(T)$. Furthermore, if $\sigma, \sigma' \in E^k(T)$ both lie on the same Θ -small geodesic, then $d_\Theta(\sigma, \sigma') = 3 \cdot d_T(m(\sigma), m(\sigma'))$.

Proof. The G -invariance of d_Θ follows from the G -invariance of both d_T and d_{E^k} . Symmetry follows from $\mathcal{M}_\Theta(\sigma, \sigma') = \mathcal{M}_\Theta(\sigma', \sigma)$. The triangle inequality holds, since we can concatenate an admissible sequence realising $d_\Theta(\sigma, \sigma')$ with an admissible sequence realising $d_\Theta(\sigma', \sigma'')$ to obtain an admissible sequence for the pair (σ, σ'') . Taking mid-, start- and endpoints of segments into account for the definition of $d_\Theta(\sigma, \sigma')$ guarantees that d_Θ is indeed a generalised metric (instead of a generalised pseudo-metric).

For two elements σ and σ' in $E^k(T)$ lying on the same Θ -small geodesic, the sequence $(\sigma, o(\sigma)), (\sigma', o(\sigma'))$ is an admissible sequence and yields $d_\Theta(\sigma, \sigma') \leq 3 \cdot d_T(m(\sigma), m(\sigma'))$. Combined use of the triangle inequality for d_T and the fact that σ, σ' lie on the same Θ -small geodesic gives that the admissible sequence $(\sigma, o(\sigma)), (\sigma', o(\sigma'))$ already realises $d_\Theta(\sigma, \sigma')$. It remains to show that d_Θ is proper:

Let $R > 0$ and $\sigma \in E^k(T)$ be given. Note that, since d_T takes values in $\mathbb{N}$, if $\sigma \neq \sigma'$, then $d_\Theta(\sigma, \sigma') \geq 1$. So, if $\sigma' \in B_R^{d_\Theta}(\sigma)$, then there is an admissible sequence $(\sigma_0, b_0), (\sigma_1, b_1), \dots, (\sigma_r, b_r)$, $r \leq R$ for the pair (σ, σ') such that $d_T(m(\sigma_i), m(\sigma_{i+1})) \leq R$ for all $0 \leq i \leq r - 1$.

Since d_{E^k} is proper, given σ_i there are only finitely many segments $\sigma \in E^k(T)$ with $d_{E^k}(\sigma_i, \sigma) \leq \Theta$. Thus, since $[m(\sigma_i), m(\sigma_{i+1})]$ has to be extended Θ -small and of length $\leq R$ (with respect to d_T), there are only finitely many possibilities for σ_1 . And given any of those possible σ_1 , there are only finitely many possibilities for σ_2 . Iterating this argument R times we conclude that there are only finitely many possibilities for $\sigma_r = \sigma'$. $\square$

2.6 The segment-flow space FS_Θ

In this section we will define a segment-flow space FS_Θ as a certain subset of the space $E^k(T) \times \Delta(T) \times \Delta(T)$. The definition given here is a slight adaptation of the coarse Θ -flow space in [Bar16, Definition 3.4] to our situation on trees.

Definition 2.20. Let $\Theta > 0$. Let $Z := \Delta(T) \times \Delta(T)$ be equipped with the subspace topology. Furthermore, let

$$FS_\Theta := \{(\sigma, \xi_-, \xi_+) \in E^k(T) \times Z \mid \sigma \subset [\xi_-, \xi_+] \text{ and } [\xi_-, \xi_+] \text{ is } \Theta\text{-small}\},$$

where the notion of a Θ -small geodesic is as in Definition 2.4.

Before we list properties of FS_Θ , we cite a definition of Bartels, which we will need briefly while applying [Bar16, Theorem 1.1] to obtain long thin covers for FS_Θ .

Definition 2.21. (see [Bar16, p. 6, before Theorem 1.1]) Let (V, d) be a generalised discrete metric space. Let $\alpha \geq 0$. A subset $S \subset V$ is α -separated if $d(x, y) \geq \alpha$ for any two distinct points $x, y \in S$. A subspace $V_0 \subset V$ has the (D, R) -doubling property if for all $\alpha \geq R$ the following holds: if $S \subset V_0$ is α -separated and contained in a 2α -ball, then the cardinality of S is at most D .

The following lemma collects the properties of $E^k(T), Z$ and FS_Θ needed to apply [Bar16, Theorem 1.1] later on.

Lemma 2.22. Let $\Theta > 0$ and let d_Θ be as in Definition 2.18. Let Z and FS_Θ be as above. Then

- a) $(E^k(T), d_\Theta)$ is a discrete countable proper generalised metric space with a proper isometric G -action.
- b) Z is a separable metric space with an action of G by homeomorphisms.
- c) FS_Θ is a closed G -invariant subspace of $E^k(T) \times Z$ (equipped with the diagonal action of G).
- d) The dimension of FS_Θ is 0.
- e) There are $D, R > 0$, independent of Θ , such that for all $(\xi_-, \xi_+) \in Z$ the subspace $E^k(T)_{(\xi_-, \xi_+)} := \{\sigma \in E^k(T) \mid (\sigma, \xi_-, \xi_+) \in FS_\Theta\}$ of $E^k(T)$ has the (D, R) -doubling property.

f) For all $(\sigma, \xi_-, \xi_+) \in FS_\Theta$, the isotropy group $G_{(\xi_-, \xi_+)} = G_{\xi_-} \cap G_{\xi_+}$ belongs to the family $\mathcal{F}_\partial \cup \mathcal{F}_{(k)}$, where $\mathcal{F}_{(k)} := \{H \leq G \mid \exists \sigma \in E^k(T) : H \leq G_{\sigma(\sigma)} \cap G_{t(\sigma)}\} \subset \mathcal{FIN}(G)$. In particular, these isotropy groups all belong to $\mathcal{F}$.

Proof. a) is immediate since d_Θ only takes values in $\mathbb{N}$ and the action of G on $E^k(T)$ has finite point stabilisers.

b) Since $\overline{T}^{obs}$ is metrisable and second-countable (see Lemma 1.30), $\Delta(T) \subset \overline{T}^{obs}$ is as well. Hence, Z is separable and metrisable.

d) By the product theorem [Nag65, Theorem II.5, p. 20]¹ and Lemma 1.33 the space $Z = \Delta(T) \times \Delta(T)$ has dimension 0. Since a discrete space V has dimension 0, $E^k(T) \times Z$ has dimension 0 by [Nag65, Theorem II.5, p. 20]. Finally, by [Nag65, Theorem II.3, p. 19], the dimension does not increase when taking subspaces and hence FS_Θ has dimension 0.

f) Since $G_{(\xi_-, \xi_+)} = G_{\xi_-} \cap G_{\xi_+}$, if $(\xi_-, \xi_+) \in \partial T \times \partial T \setminus \text{diag}$, then $G_{(\xi_-, \xi_+)} \in \mathcal{F}_\partial$ by definition. If at least one of the points ξ_-, ξ_+ lies in T , then, by definition of FS_Θ , the group $G_{\xi_-} \cap G_{\xi_+}$ is contained in the pointwise stabiliser G_σ of some $\sigma \in E^k(T)$, and hence is finite. It remains to show c) and e).

c) Since $E^k(T) \times Z$ is metrisable, we can work with sequences. Moreover, without loss of generality any convergent sequence has the form

$$(\sigma, \xi_{-,n}, \xi_{+,n}) \rightarrow (\sigma, \xi_-, \xi_+) \in E^k(T) \times Z,$$

since $E^k(T)$ is discrete. Now, if all $(\sigma, \xi_{-,n}, \xi_{+,n})$ lie in FS_Θ , then $\sigma \subset [\xi_{-,n}, \xi_{+,n}]$ holds for all $n \in \mathbb{N}$. Hence, the topology of $\Delta(T)$ forces σ to lie on $[\xi_-, \xi_+]$ (though $\sigma = [\xi_-, \xi_+]$ is possible). The geodesic $[\xi_-, \xi_+]$ is Θ -small, since the geodesics $[\xi_{-,n}, \xi_{+,n}]$ are and any finite subsegment of $[\xi_-, \xi_+]$ is eventually contained in some $[\xi_{-,N}, \xi_{+,N}]$. G -invariance of FS_Θ is immediate, since the notion of a Θ -small geodesic is G -invariant.

e) We will show that for all (ξ_-, ξ_+) the subspace $E^k(T)_{(\xi_-, \xi_+)}$ has the $(5, 3)$ -doubling property: So let $\alpha \geq R = 3$. First, note that $E^k(T)_{(\xi_-, \xi_+)}$ is empty if the geodesic $[\xi_-, \xi_+]$ is not Θ -small or is of length $\leq k - 1$. Otherwise $E^k(T)_{(\xi_-, \xi_+)}$ is equal to $E^k(T) \cap [\xi_-, \xi_+]$. In this case, we can take midpoints of segments to obtain a bijection between $E^k(T)_{(\xi_-, \xi_+)}$ and (a subset of) $\mathbb{Z}$. Since $d_\Theta(\sigma, \sigma') = 3 \cdot d_T(m(\sigma), m(\sigma'))$ for all σ, σ' on $[\xi_-, \xi_+]$, any set of the form $B_{2\alpha}(\tilde{\sigma}) \cap E^k(T)_{(\xi_-, \xi_+)}$ consists of at most $\frac{4}{3}\alpha + 1$ elements of $E^k(T)$. Furthermore, if $\sigma, \sigma' \in E^k(T)_{(\xi_-, \xi_+)}$ belong to an α -separated subset, then $d_T(m(\sigma), m(\sigma')) \geq \frac{\alpha}{3}$ follows from $d_\Theta(\sigma, \sigma') = 3 \cdot d_T(m(\sigma), m(\sigma'))$. Thus, there are at most

$$\frac{\frac{4}{3}\alpha + 1}{\frac{\alpha}{3}} \leq 5 =: D$$

elements in any α -separated subset of $E^k(T)_{(\xi_-, \xi_+)}$ which is contained in a 2α -ball. $\square$

¹This and the following results referenced are formulated for the strong inductive dimension. However, recall that for separable metrisable spaces, the small inductive dimension coincides with the covering dimension (and the strong inductive dimension) [Nag65, Theorem IV.1, p. 90].

Next, we define a family of maps from $G \times_{\Theta} \partial T$ to FS_{Θ} that formalise the idea of letting an element σ of $E^k(T)$ ‘flow along a Θ -small geodesic $[a(g, \xi), \xi]$ ’. Morally, this family of maps could be thought of as a partial flow on FS_{Θ} .

Definition 2.23. Let $\tau \in \mathbb{N}$, $\Theta > \frac{k}{2}$ and let $G \times_{\Theta} \partial T$ be as in Notation 2.5. We define

$$\phi_{\Theta, \tau} : G \times_{\Theta} \partial T \longrightarrow FS_{\Theta}, \quad (g, \xi) \mapsto (\sigma_{\tau}(g, \xi), a(g, \xi), \xi),$$

where $\sigma_{\tau}(g, \xi)$ is the unique geodesic segment of length k on $[a(g, \xi), \xi]$ such that

$$d_T(a(g, \xi), m(\sigma_{\tau}(g, \xi))) = \tau.$$

Lemma 2.24. $\phi_{\Theta, \tau}$ is continuous and G -equivariant.

Proof. G -equivariance is clear, since the action of G on T is simplicial. Let $(g, \xi_n) \in G \times_{\Theta} \partial T$ be a sequence converging to some $(g, \xi) \in G \times_{\Theta} \partial T$. Since $a(g, \eta) \in \{gw_0, gw'_0\}$ for arbitrary η , we can assume without loss of generality that $a(g, \xi_n) = a(g, \xi)$ for all $n \in \mathbb{N}$. The convergence of $\xi_n \rightarrow \xi$ then already implies that $\sigma_{\tau}(g, \xi_n) = \sigma_{\tau}(g, \xi)$ for all but finitely many n . $\square$

2.7 Wide covers for $G \times_{\Theta} \partial T$

Given some threshold $\Theta > 0$, this section provides an open G -invariant $\mathcal{F}_{\partial} \cup \mathcal{F}_{(k)}$ -cover of $G \times_{\Theta} \partial T$ that is wide (for all of $G \times_{\Theta} \partial T$). Again, this is done analogously to [Bar16]. We start by using [Bar16, Theorem 1.1] to obtain a family of covers for the flow space FS_{Θ} .

Corollary 2.25. Let $\Theta > 0$. Let FS_{Θ} be as before and let D and R be as in Lemma 2.22. Then there is $N \in \mathbb{N}$ independent of Θ , such that for any $\alpha > 0$ there is an open G -invariant $\mathcal{F}_{\partial} \cup \mathcal{F}_{(k)}$ -cover $\mathcal{W}_{\Theta, \alpha}$ of FS_{Θ} satisfying

a) $\dim \mathcal{W}_{\Theta, \alpha} \leq N$,

b) for every $(\sigma, \xi_-, \xi_+) \in FS_{\Theta}$ there is $W \in \mathcal{W}_{\Theta, \alpha}$ such that

$$(B_{\alpha}^{d_{\Theta}}(\sigma) \times \{(\xi_-, \xi_+)\}) \cap FS_{\Theta} \subset W.$$

Proof. This is the conclusion of [Bar16, Theorem 1.1]. That the assumptions of this theorem are satisfied by FS_{Θ} follows from Lemma 2.22. $\square$

Pulling back these covers along $\phi_{\Theta, \tau}$ gives the following corollary.

Corollary 2.26. There is N such that for all $\alpha > 0$, $\Theta > 0$ and $\tau \in \mathbb{N}$ the set

$$\phi_{\Theta, \tau}^{-1}(\mathcal{W}_{\Theta, \alpha}) := \{\phi_{\Theta, \tau}^{-1}(W) \mid W \in \mathcal{W}_{\Theta, \alpha}\}$$

is an open G -invariant $\mathcal{F}_{\partial} \cup \mathcal{F}_{(k)}$ -cover of $G \times_{\Theta} \partial T$ of dimension $\leq N$.

Before we show that among these covers we can find one which is wide for $G \times_{\Theta} \partial T$, we will state two technical assertions.

Lemma 2.27. *Let $\alpha > 0$. Let $v \in V(T)$ and $g_{\tau}v \rightarrow \xi \in \partial T$ for $\tau \rightarrow \infty$. Then, for all $h \in B_{\alpha}(1)$, the sequence $g_{\tau}hv$ also converges to ξ .*

Proof. Fix a basepoint $b \in T$. So $[b, g_{\tau}v]$ converges uniformly on compacta to $[b, \xi]$. Let $C := \max\{d_T(v, hv) \mid h \in B_{\alpha}(1)\}$. Since $d_T(g_{\tau}v, g_{\tau}hv) = d_T(v, hv) \leq C$ for all τ , the sequence $[b, g_{\tau}hv]$ converges uniformly on compacta to $[b, \xi]$ if and only if the sequence $[b, g_{\tau}v]$ does. $\square$

Lemma 2.28. *Let $\Theta > 0$. Let $\xi_{-,n} \rightarrow \xi_-$ and $\xi_{+,n} \rightarrow \xi_+$ be convergent sequences in $\Delta_+(T)$. Furthermore, assume that there is a segment $\sigma \in E^k(T)$ such that σ lies on all geodesics $[\xi_{-,n}, \xi_{+,n}]$ and that for all $r > 0$ the set $B_r^{d_T}(\sigma) := \{z \in T \mid d_T(z, \sigma) \leq r\}$ contains only finitely many of the $\xi_{-,n}, \xi_{+,n}$. If the geodesics $[\xi_{-,n}, \xi_{+,n}]$ are Θ -small, then $\xi_-, \xi_+ \in \partial T$ and $[\xi_-, \xi_+]$ is Θ -small.*

Proof. Assume that $\xi_+ \notin \partial T$. If $\xi_{+,n} = \xi_+$ for all $n \geq n_0$, there is—in contradiction to the assumptions of the lemma—some $r > 0$ such that $B_r^{d_T}(\sigma)$ contains infinitely many of $\xi_{+,n}$. So, for $\xi_{+,n} \rightarrow \xi_+ \in V(T)$ to hold, the (union of the) generalised geodesics $[\xi_+, \xi_{+,n}]$ must contain infinitely many of the edges incident to ξ_+ . Since d_{E^k} is proper, this contradicts the other assumption of the lemma, that all $[o(\sigma), \xi_{+,n}]$ are Θ -small. The argument for $\xi_- \in \partial T$ is the same and the geodesic $[\xi_-, \xi_+]$ is Θ -small since all geodesics $[\xi_{-,n}, \xi_{+,n}]$ are. $\square$

Proposition 2.29. *There is $N \in \mathbb{N}$ such that for all $\alpha > 0$ and all $\Theta > 0$ there is a G -invariant collection $\mathcal{U}_{\Theta, \alpha}$ of open $\mathcal{F}_{\partial} \cup \mathcal{F}_{(k)}$ -subsets of $G \times \Delta(T)$ satisfying*

- a) *the order of $\mathcal{U}_{\Theta, \alpha}$ is at most N ;*
- b) *for all $(g, \xi) \in G \times_{\Theta} \partial T$ there is $U \in \mathcal{U}_{\Theta, \alpha}$ such that $B_{\alpha}(g) \times \{\xi\} \subset U$.*

Proof. (following the proof of [Bar16, Proposition 3.2]) Let N be as in Corollary 2.25 and Corollary 2.26. Let $\alpha, \Theta > 0$ be given. Recall that L_0 is the finite subtree of T spanned by the vertices $\{hw_0, hw'_0 \mid h \in B_{\alpha}(1)\}$. Let $C = C(\alpha)$ be the diameter of L_0 with respect to d_T . (In particular, for any $\xi \in \Delta(T)$, $h \in B_{\alpha}(1)$ and $g \in G$ the distance of $a(g, \xi)$ to $a(gh, \xi)$ is bounded by C .) Set $\alpha' := 3C$ and let $\mathcal{W}_{\Theta, \alpha'}$ be as in Corollary 2.25.

We will show that there is $\tau_0 \in \mathbb{N}$ such that $\phi_{\Theta, \tau}^{-1}(\mathcal{W}_{\Theta, \alpha'})$ satisfies b). Once we have proven this, we can thicken $\phi_{\Theta, \tau_0}^{-1}(\mathcal{W}_{\Theta, \alpha'})$ by Lemma 1.15 to a G -invariant collection $\mathcal{U}_{\Theta, \alpha}$ of open $\mathcal{F}$ -subsets of $G \times \Delta(T)$ that is still wide for $G \times_{\Theta} \partial T$ (i.e. still satisfies b)). Since all $\phi_{\Theta, \tau}^{-1}(\mathcal{W}_{\Theta, \alpha'})$ satisfy a) by Corollary 2.26, the collection $\mathcal{U}_{\Theta, \alpha}$ will satisfy a) as well by Lemma 1.15.

To show that for a given $\alpha, \Theta > 0$ there is a $\phi_{\Theta, \tau_0}^{-1}(\mathcal{W}_{\Theta, \alpha'})$ that is wide, start by assuming the contrary.

Assumption: For all $\tau \in \mathbb{N}$ there is $(g_\tau, \xi_\tau) \in G \times_\Theta \partial T$ such that for all $W \in \mathcal{W}_{\Theta, \alpha'}$

$$B_\alpha(g_\tau) \times \{\xi_\tau\} \not\subset \phi_{\Theta, \tau}^{-1}(W) .$$

Applying $\phi_{\Theta, \tau}$ to (g_τ, ξ_τ) we obtain a segment $\sigma_\tau(g_\tau, \xi_\tau)$ of length k on $[a(g_\tau, \xi_\tau), \xi_\tau]$. Let σ_1 be the first segment of length k on $[w_0, w'_0]$ and σ_2 be the last segment of length k on $[w_0, w'_0]$ (those segments exist since $[w_0, w'_0]$ has length $\geq 5k$). Since $[a(g_\tau, \xi_\tau), \xi_\tau]$ contains $[a(g_\tau, \xi_\tau), gm_\varepsilon]$, the first segment of length k on $[a(g_\tau, \xi_\tau), \xi_\tau]$ is either $g_\tau\sigma_1$ or $g_\tau\sigma_2$. Furthermore, since all $[a(g_\tau, \xi_\tau), \xi_\tau]$ are Θ -small and their respective first segments of length k lie in at most two different G -orbits of $G \curvearrowright E^k(T)$, by our special construction of d_{E^k} (cf. Lemma 2.3), it follows that the segments $\sigma_\tau(g_\tau, \xi_\tau)$ can only belong to a finite number of G -orbits of $G \curvearrowright E^k(T)$. Since $\phi_{\Theta, \tau}^{-1}(\mathcal{W}_{\Theta, \alpha'})$ is a G -invariant collection we can assume (after passing to a subsequence) that there is a segment $\sigma \in E^k(T)$ such that $\sigma = \sigma_\tau(g_\tau, \xi_\tau)$ for all τ .

Since $\Delta_+(T)$ is compact (cf. Lemma 1.31), by passing to a further subsequence (twice) we can enforce the existence of $\xi_+ = \lim_\tau \xi_\tau$ and $\xi_- = \lim_\tau a(g_\tau, \xi_\tau)$ in $\Delta_+(T)$. Since τ increases, no $B_r^{d_T}(\sigma)$ contains infinitely many of the $a(g_\tau, \xi_\tau)$. Moreover, by assumption, all ξ_τ lie in ∂T . Hence, by Lemma 2.28, we obtain that ξ_- and ξ_+ both lie in ∂T and that $[\xi_-, \xi_+]$ is Θ -small. Furthermore, σ lies on $[\xi_-, \xi_+]$. In other words, $(\sigma, \xi_-, \xi_+) \in FS_\Theta$.

Now, by Corollary 2.25, there is $W_0 \in \mathcal{W}_{\Theta, \alpha'}$ such that

$$(B_{\alpha'}^{d_\Theta}(\sigma) \times \{(\xi_-, \xi_+)\}) \cap FS_\Theta \subset W_0.$$

As d_Θ is proper, the ball $B_{\alpha'}^{d_\Theta}(\sigma)$ is finite and so is $M := B_{\alpha'}^{d_\Theta}(\sigma) \cap [\xi_-, \xi_+]$. Since $\xi_-, \xi_+ \in \partial T$ and W_0 is open, there are disjoint open neighbourhoods U_- and U_+ in $\Delta(T)$ of ξ_- and ξ_+ , respectively, such that

$$(M \times U_- \times U_+) \cap FS_\Theta \subset W_0.$$

Without loss of generality we can assume that U_+ is of the form $M(\xi_+, a)$ for some $a \in T$ with $d_T(t(\sigma), a) > C$ (the purpose of this technical assumption will only be used at the very end of this proof).

By Lemma 2.27, we still have $\lim_\tau a(g_\tau h, \xi_\tau) = \xi_-$ for all $h \in B_\alpha(1)$. So we can find $\tau_{0,h}$ such that $a(g_\tau h, \xi_\tau) \in U_-$ and $\xi_\tau \in U_+$ for all $\tau \geq \tau_{0,h}$. Since $B_\alpha(1)$ is finite and $B_\alpha(g_\tau) = g_\tau B_\alpha(1)$, there is a τ_0 such that $a(g, \xi_\tau) \in U_-$ and $\xi_\tau \in U_+$ for all $\tau \geq \tau_0$ and $g \in B_\alpha(g_\tau)$. Since M is finite as well, we can (if necessary) enlarge τ_0 further to obtain that all elements of M already lie on $[a(g_{\tau_0}, \xi_{\tau_0}), \xi_{\tau_0}]$. We now claim that there is some $\tau \geq \tau_0$ such that

$$g_\tau B_\alpha(1) \times \{\xi_\tau\} = B_\alpha(g_\tau) \times \{\xi_\tau\} \subset \phi_{\Theta, \tau}^{-1}(W_0),$$

which would be a contradiction to the assumption we started with. To prove this claim it is sufficient to find τ such that

$$(\sigma_\tau(g_\tau h, \xi_\tau), a(g_\tau h, \xi_\tau), \xi_\tau) \in M \times U_- \times U_+$$

for all $h \in B_\alpha(1)$. By the way τ_0 was defined, it is guaranteed that $\xi_\tau \in U_+$ and $a(g_\tau h, \xi_\tau) \in U_-$ for all $\tau \geq \tau_0$. So it only remains to find $\tau \geq \tau_0$ large enough such that $\sigma_\tau(g_\tau h, \xi_\tau) \in M$ holds for all $h \in B_\alpha(1)$.

Let v_h be the first vertex on $[a(g_\tau h, \xi_\tau), \xi_\tau]$ that also lies on $[a(g_\tau, \xi_\tau), \xi_\tau]$, i. e.

$$[a(g_\tau h, \xi_\tau), \xi_\tau] \cap [a(g_\tau, \xi_\tau), \xi_\tau] = [v_h, \xi_\tau].$$

Since $B_\alpha(1)$ is finite, there is a vertex $v = v_{h_0}$ which—among the v_h —has maximal distance to $a(g_\tau, \xi_\tau)$. In particular, $[v, \xi_\tau]$ is contained in $[a(g_\tau h, \xi_\tau), \xi_\tau]$ for all $h \in B_\alpha(1)$. Note that, since all $a(g_\tau h, \xi_\tau)$ lie in U_- , all v_h lie in U_- as well. In particular, v lies in U_- . Moreover, the vertex v lies on the geodesic $[a(g_\tau h_0, \xi_\tau), a(g_\tau h, \xi_\tau)]$ for all $h \in B_\alpha(1)$. Since the length of $[a(g_\tau h_0, \xi_\tau), a(g_\tau h, \xi_\tau)]$ is bounded by C , the vertex v has distance at most C to any of the $a(g_\tau h, \xi_\tau)$. Hence, (by choosing $\tau \geq \tau_0$ large enough) we can force $\sigma_\tau(g_\tau h, \xi_\tau)$ to lie on $[v, \xi_\tau]$ for all $h \in B_\alpha(1)$. Since $d_T(a(g_\tau, \xi_\tau), m(\sigma_\tau(g_\tau, \xi_\tau))) = \tau$, the bound C and the triangle inequality for d_T imply

$$\tau + C \leq d_T(a(g_\tau, \xi_\tau), m(\sigma_\tau(g_\tau h, \xi_\tau))) \geq \tau - C.$$

Both $\sigma_\tau(g_\tau h, \xi_\tau)$ and $\sigma_\tau(g_\tau, \xi_\tau)$ lie on $[v, \xi_\tau]$ and it follows that

$$d_T(m(\sigma_\tau(g_\tau h, \xi_\tau)), m(\sigma_\tau(g_\tau, \xi_\tau))) \leq C.$$

Since the geodesic $[a(g_\tau, \xi_\tau), \xi_\tau]$ is Θ -small and contains both $\sigma_\tau(g_\tau h, \xi_\tau)$ and $\sigma_\tau(g_\tau, \xi_\tau)$, by definition of d_Θ it follows that

$$d_\Theta(\sigma_\tau(g_\tau h, \xi_\tau), \sigma_\tau(g_\tau, \xi_\tau)) = 3 \cdot d_T(m(\sigma_\tau(g_\tau h, \xi_\tau)), m(\sigma_\tau(g_\tau, \xi_\tau))) \leq 3C = \alpha'.$$

Hence, $\sigma_\tau(g_\tau h, \xi_\tau) \in B_{\alpha'}^{d_\Theta}(\sigma_\tau(g_\tau, \xi_\tau))$ for all $h \in B_\alpha(1)$. Recall that $\sigma = \sigma_\tau(g_\tau, \xi_\tau)$ and $U_+ = M(\xi_+, a)$ where a satisfies $d_T(t(\sigma), a) \geq C$. Therefore, $\sigma_\tau(g_\tau h, \xi_\tau)$ indeed lies on $[\xi_-, \xi_+]$ and is contained in $B_{\alpha'}^{d_\Theta}(\sigma)$, so it is contained in M . $\square$

2.8 Finitely $\mathcal{F}$ -amenability of $G \curvearrowright \Delta(T)$

The proof of Theorem 2.2 now reduces to formally combining the wide covers constructed in Proposition 2.9 and Proposition 2.29 for the sets $G \times \Delta(T) \setminus G \times_\Theta \partial T$ and $G \times_\Theta \partial T$, respectively.

Theorem 2.2. *Let G be a countable group and T be a countable tree. Let $G \curvearrowright T$ be a (not necessarily strictly or uniformly) acylindrical action and let $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_\partial$ (as in Notation 2.1). Then the action $G \curvearrowright \Delta(T)$ of G on the Bowditch boundary of T is finitely $\mathcal{F}$ -amenable.*

Proof. Recall first, that the theorem is trivial if $G \curvearrowright T$ has a global fixed point. Secondly, by substituting T with its barycentric subdivision, we can assume without loss of generality

that $k \geq 2$ is even and that the set $V_0(T) = V(T) \setminus V_\infty(T)$ is unbounded (with respect to d_T). So, we are indeed free to use all results from the previous sections.

Let $N_1 := k + 1$ (which is the N from Proposition 2.9) and let N_2 be the N from Proposition 2.29. Set $N := N_1 + N_2 + 1$ and let $\alpha > 0$. By Lemma 2.8 and Proposition 2.9, there are thresholds $\Theta_i, -1 \leq i \leq N_1$ such that the G -invariant collection of open $\mathcal{F}_T$ -subsets

$$\bigcup_{i=0}^{N_1} \mathcal{W}_{\Theta_i, \alpha} = \bigcup_{i=0}^{N_1} \{W(v, \Theta_i) \mid v \in V(T)\}$$

is wide for $G \times \Delta(T) \setminus G \times_{\Theta_{-1}} \partial T$. Let $\mathcal{U}_{\Theta_{-1}, \alpha}$ be the G -invariant collection of open $\mathcal{F}_\partial \cup \mathcal{F}_{(k)}$ -subsets obtained from Proposition 2.29 (for $\Theta = \Theta_{-1}$). In particular, $\mathcal{U}_{\Theta_{-1}, \alpha}$ is wide for $G \times_{\Theta_{-1}} \partial T$. Therefore, the G -invariant collection of open $\mathcal{F}$ -subsets

$$\mathcal{U}_\alpha := \bigcup_{i=0}^{N_1} \mathcal{W}_{\Theta_i, \alpha} \cup \mathcal{U}_{\Theta_{-1}, \alpha},$$

is wide for $G \times \Delta(T)$ and $\mathcal{U}_\alpha$ is a cover for $G \times \Delta(T)$. Since the order of each of the $\mathcal{W}_{\Theta_i, \alpha}$ is 0 (see Lemma 2.8), the order of $\bigcup_{i=0}^{N_1} \mathcal{W}_{\Theta_i, \alpha}$ is at most N_1 . The order of $\mathcal{U}_{\Theta_{-1}, \alpha}$ is at most N_2 (see Proposition 2.29 a)), and thus the dimension of $\mathcal{U}_\alpha$ is at most $N_1 + N_2 + 1$. $\square$

Chapter 3

The main results

The purpose of this chapter is to give the main results of this thesis and, in particular, prove the statements given in the introduction. Section 3.1 contains the results on the Farrell–Jones Conjecture for acylindrical group actions on trees. Based on these findings, in Section 3.2 we generalise a result by Lafont and Ortiz [LO09] on Waldhausen Nil-groups associated to an amalgamated free product acting acylindrically on its Bass-Serre tree.

3.1 The Farrell–Jones Conjecture and acylindrical actions on trees

The first main result of this thesis is the following.

Theorem 3.1. *Let G be a group acting acylindrically on a simplicial tree T and let the family $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_\partial$ be as in Notation 2.1. Let $\mathcal{F}_2$ be the family of subgroups of G that contain a group in $\mathcal{F}$ as a subgroup of index ≤ 2 . Then*

- a) G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}$;*
- b) G satisfies the L -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}_2$.*

The major work towards this theorem was already done in Chapter 2, where we proved that G acts finitely $\mathcal{F}$ -amenable on $\Delta(T)$. We would like to use the geometric conditions from Subsection 1.1.3 to conclude the above theorem. However, $\Delta(T)$ is not a contractible space, so we have to extend Theorem 2.2 to $\overline{T}^{obs}$ first. Afterwards, inheritance properties of the Farrell–Jones Conjecture and existence of minimal G -invariant subtrees will suffice to show the theorem.

Corollary 3.2. *Let G be a countable group and T be countable tree. Let $G \curvearrowright T$ be an acylindrical action and let $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_\partial$. Then the action $G \curvearrowright \overline{T}^{obs}$ of G on the Bowditch compactification of T is finitely $\mathcal{F}$ -amenable.*

Proof. By Theorem 2.2, there is N' such that the action of G on the Bodwitch boundary $\Delta(T)$ is N' - $\mathcal{F}$ -amenable. Let $N := N' + 2$. To produce, for all $\alpha > 0$, an open G -invariant $\mathcal{F}$ -cover of $G \times \bar{T}^{obs}$ that is wide and of dimension at most N , we first tend to covering $G \times \bar{T}^{obs} \setminus G \times \Delta(T)$.

For a moment we consider T with the path-metric topology and define two kinds of open sets of T . Let $\mathcal{I}$ be all open edges of T and $\mathcal{B}$ be all open balls of radius ε (i.e. some fixed small positive number) around vertices of T . Then, both $\mathcal{I}$ and $\mathcal{B}$ are G -invariant collections of open $\mathcal{F}$ -subsets of T (with the path-metric topology). Since the observers' topology coincides with the path-metric topology on finite subtrees, it follows that $\mathcal{V}' := \{U \setminus V_\infty(T) \mid U \in \mathcal{I} \cup \mathcal{B}\}$ consists of open sets of $\bar{T}^{obs}$. Moreover, $\mathcal{V}'$ is a G -invariant collection of open $\mathcal{F}$ -subsets of $\bar{T}^{obs}$, since $V_\infty(T)$ is G -invariant. As $\mathcal{I} \cup \mathcal{B}$ is of order 1, so is $\mathcal{V}'$. Define $\mathcal{V} := \{G \times U' \mid U' \in \mathcal{V}'\}$. This is a G -invariant collection of open $\mathcal{F}$ -subsets of $G \times \bar{T}^{obs}$ which still has order 1. For $\xi \in \bar{T}^{obs} \setminus \Delta(T)$ there is an $U' \in \mathcal{V}'$ with $\xi \in U'$. Thus—independent of $\alpha > 0$ —for all $(g, \xi) \in G \times (\bar{T}^{obs} \setminus \Delta(T))$ there is an element of $\mathcal{V}$ that contains $B_\alpha(g) \times \{\xi\}$. In other words, $\mathcal{V}$ is wide for $G \times (\bar{T}^{obs} \setminus \Delta(T))$.

Now, let $\alpha > 0$. By Theorem 2.2, there is an open G -invariant $\mathcal{F}$ -cover $\mathcal{U}_\alpha$ of $G \times \Delta(T)$ of dimension at most N' which is wide. Let $\mathcal{U}_S^+$ be the result of the thickening process of Lemma 1.15. So $\mathcal{U}_S^+$ is a G -invariant collection of open $\mathcal{F}$ -subsets of $G \times \bar{T}^{obs}$ that is wide for $G \times \Delta(T)$ and has order at most N' . Finally, defining $\mathcal{V}_S := \mathcal{V} \cup \mathcal{U}_S^+$ gives the desired open G -invariant $\mathcal{F}$ -cover of $G \times \bar{T}^{obs}$ which is wide for all of $G \times \bar{T}^{obs}$ and is of dimension at most $N = N' + 2$. $\square$

Proof of Theorem 3.1. For a finitely generated subgroup G' of G let $\mathcal{F}'$ denote the family $\mathcal{F} \cap G'$. By Lemma 1.6, we know that a) (resp. b)) holds if the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{F}'$ (resp. $\mathcal{F}'_2$) holds for all finitely generated subgroups G' of G . Hence, it suffices to prove the theorem for G finitely generated.

By Lemma 1.16, there is a minimal G -invariant subtree $T_{\min}$ of T such that G acts cocompactly on $T_{\min}$. Moreover, $G \curvearrowright T_{\min}$ is acylindrical. Denote by $\mathcal{F}(G, T_{\min})$ the family of subgroups of G obtained by applying Notation 2.1 to the action $G \curvearrowright T_{\min}$. By Corollary 1.5, we know that a) (resp. b)) holds if the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\mathcal{F}(G, T_{\min})$ (resp. $\mathcal{F}(G, T_{\min})_2$) holds. Hence, we can assume without loss of generality that G and T are countable. By Corollary 3.2, the action of G on $\bar{T}^{obs}$ is finitely $\mathcal{F}$ -amenable. Finally, since $\bar{T}^{obs}$ is a compact contractible controlled 1-dominated metric space (see Lemma 1.30, Lemma 1.31 and Lemma 1.36) the claim follows by Theorem 1.12. $\square$

Next, we examine the family $\mathcal{F}_\partial$ more closely. It is known that $\mathcal{F}_\partial \subset \mathcal{VCY}$ (see for instance [LO09, Claim 2]) and in the next lemma we refine this finding. Since its proof is not particularly involved, the author expects the lemma to be well-known, but could not find a reference for this particular statement either.

Lemma 3.3. *Let G be a group acting acylindrically on a simplicial tree T . Then each element in $\mathcal{F}_\partial$ is either finite or virtually cyclic of type I (i.e. surjects onto $\mathbb{Z}$).*

Proof. For $g \in G$, if the translation length $\|g\|$ of g is > 0 , denote by C_g its translation axis. If $\|g\| = 0$, denote by C_g the subtree of T which is pointwise fixed by g .

Let $\xi \in \partial T$. Then for all $g \in G_\xi$ the set C_g has unbounded intersection with each geodesic ray ρ representing ξ . ($C_g \cap \rho$ being empty or bounded immediately gives a contradiction to $g \in G_\xi$, since T is a tree; cf. Figure 3.1 for an exemplary case).

Figure 3.1: Bounded intersection of C_g and ρ implies $g \notin G_\xi$.

Let $(\xi_1, \xi_2) \in \partial T \times \partial T \setminus \text{diag}$. Then any $g \in G_{\xi_1} \cap G_{\xi_2}$ must satisfy $[\xi_1, \xi_2] \subset C_g$ (and equality holds if $\|g\| > 0$). Hence, $G_{\xi_1} \cap G_{\xi_2}$ can only contain finitely many elements with translation length 0. In particular, if the group $G_{\xi_1} \cap G_{\xi_2}$ consists only of elements with translation length 0, this group is finite. So from now on we assume that there is at least one element $\bar{g} \in G_{\xi_1} \cap G_{\xi_2}$ with $\|\bar{g}\| > 0$. Since $\|g\| \in \mathbb{N}$ there is $g_0 \in G_{\xi_1} \cap G_{\xi_2}$ with $\|g_0\| > 0$ minimal.

The translation length $\|h\|$ of any $h \in G_{\xi_1} \cap G_{\xi_2}$ with $\|h\| > 0$ must be divisible by $\|g_0\|$: Assume $\|g_0\|$ does not divide $\|h\|$. Then there are integers α, β such that $\|g_0\| > \gcd(\|g_0\|, \|h\|) = \alpha\|g_0\| + \beta\|h\|$. Hence, the element $g_0^\alpha h^\beta$ acts as a shift on $[\xi_1, \xi_2]$ and therefore lies in $G_{\xi_1} \cap G_{\xi_2}$. Since $g_0^\alpha h^\beta$ has translation length less than $\|g_0\|$, this is a contradiction.

Let $\{g_1, \dots, g_K\}$ be the set of all elements in $G_{\xi_1} \cap G_{\xi_2}$ with translation length 0. If $h \in G_\xi$ and $\|h\| = 0$, then $h = g_i$ for some $1 \leq i \leq K$ and, in particular, $h = g_i g_0^0 \in g_i \langle g_0 \rangle$. If $h \in G_\xi$ with $\|h\| = \alpha\|g_0\| > 0$, then one of the two elements $h g_0^{\pm \alpha}$ fixes $[\xi_1, \xi_2]$ pointwise and thus must lie in $\{g_1, \dots, g_K\}$. In other words, we have $h = g_i g_0^\alpha$ for some $\alpha \in \mathbb{Z}$ and $1 \leq i \leq K$. Hence, $\mathbb{Z} = \langle g_0 \rangle$ is of finite index in $G_{\xi_1} \cap G_{\xi_2}$. Moreover, sending $g_i g_0^\alpha$ to g_0^α gives the desired surjection onto $\mathbb{Z}$. $\square$

Inheritance properties of the Farrell–Jones Conjecture allow us to deduce the conjecture relative to $\mathcal{VCYC}$ if it is known (relative to $\mathcal{VCYC}$) for all groups in $\mathcal{F}$ (resp. $\mathcal{F}_2$).

Corollary 3.4. *Let G be a group acting acylindrically on a simplicial tree T . If all vertex stabilisers of $G \curvearrowright T$ satisfy the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$, then G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$.*

Proof. By Theorem 3.1, G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}_T \cup \mathcal{F}_\partial$. Hence, by Lemma 3.3 and Corollary 1.5, G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}_T \cup \mathcal{VCYC}$. If $H \in \mathcal{F}_T$, then H satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$ by assumption and Theorem 1.3. If $H \in \mathcal{VCYC}$, H

satisfies the K -theoretic Farrell–Jones Conjecture trivially. Thus, by Theorem 1.4, the claim follows. $\square$

The L -theoretic case works analogously, but we have to take overgroups of index 2 of vertex stabilisers of $G \curvearrowright T$ into account.

Corollary 3.5. *Let G be a group acting acylindrically on a simplicial tree T . Assume that all (subgroups of G that are) overgroups of index at most 2 of vertex stabilisers of $G \curvearrowright T$ satisfy the L -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$. Then G satisfies the L -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$.*

Proof. By Theorem 3.1, G satisfies the L -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}_2 = (\mathcal{F}_T \cup \mathcal{F}_\partial)_2$. Since the family of virtually cyclic subgroups of G is closed under taking overgroups of finite index, the group G satisfies the L -theoretic Farrell–Jones Conjecture relative to $(\mathcal{F}_T)_2 \cup \mathcal{VCYC}$ by Lemma 3.3 and Corollary 1.5. If $H \in (\mathcal{F}_T)_2$, then H satisfies the L -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$ by assumption and Theorem 1.3. If $H \in \mathcal{VCYC}$, H satisfies the L -theoretic Farrell–Jones Conjecture trivially. Thus, by Theorem 1.4, the claim follows. $\square$

These two corollaries have a counterpart for the Farrell–Jones Conjecture with finite wreath products: A group G is said to satisfy the *Farrell–Jones Conjecture with finite wreath products relative to a family $\mathcal{F}$* if for all finite groups F the wreath product $G \wr F$ satisfies the Farrell–Jones Conjecture relative to $\mathcal{F}$. Recalling that we proved Theorem 3.1 basically by showing that the group G involved is strongly transfer reducible over $\mathcal{F}_T \cup \mathcal{F}_\partial$ (cf. the proof of Theorem 1.12), we can obtain the following variant of the last two corollaries by using [BLRR14, Theorem 5.1] and the transitivity principle.

Corollary 3.6. *Let G be group acting acylindrically on a simplicial tree T . Assume that all vertex stabilisers of $G \curvearrowright T$ satisfy the Farrell–Jones Conjecture with finite wreath products relative to $\mathcal{VCYC}$. Then G satisfies the Farrell–Jones Conjecture with finite wreath products relative to $\mathcal{VCYC}$.*

For an amalgamated free product $G = A *_C B$, where A and B are hyperbolic and C is quasi-isometrically embedded in both A and B and is malnormal in one of the factors, the Combination Theorem of Bestvina and Feighn [BF92, BF96] implies that G is hyperbolic, and hence, the Farrell–Jones Conjecture relative to $\mathcal{VCYC}$ is known for G . For an amalgamated free product $G = A *_C B$ with A and B abelian, the last conclusion follows by [GMR15]. In both cases combining knowledge about A and B to obtain the Farrell–Jones Conjecture for G only works because A and B are the same type of groups.

We conclude this section with two examples illustrating that Corollary 3.4 does not have this restriction.

Example 3.7. Let A be a torsion-free hyperbolic group. Then, by [Kap99, Theorem C], there is a malnormal subgroup $C \cong F_2$ of A . Let $B := \mathrm{SL}_n(\mathbb{Z})$ (or $\mathrm{GL}_n(\mathbb{Z})$) for some $n \geq 3$. By [Hum88, Theorem 1.1], we find a subgroup C' of B isomorphic to F_n . So we

can form the group $G := A *_C B$ and it acts acylindrically on its associated Bass-Serre tree by Lemma 1.20.

The Farrell–Jones Conjecture (relative to $\mathcal{VCYC}$) with finite wreath products holds for any hypbolic group by [BLR08a, Main Theorem], [BL12a, Theorem B(ii)] and [Weg15, Proposition 2.20, Example 2.21]. By [BLRR14, General Theorem], the Farrell–Jones Conjecture (relative to $\mathcal{VCYC}$) with finite wreath products holds for $GL_n(\mathbb{Z})$ as well. So, by the above corollary, the K - and L -theoretic Farrell–Jones Conjecture with finite wreath products (relative to $\mathcal{VCYC}$) hold for G .

Example 3.8. Let G_0 and G be as in Example 1.24 and let $G' = \operatorname{colim}_{n \in \mathbb{N}} SL(n, \mathbb{Z})$. Then G' satisfies the Farrell–Jones Conjecture (relative to $\mathcal{VCYC}$) with finite wreath products by [BLRR14, General Theorem] and [Weg15, Proposition 2.17].

To see that G satisfies the Farrell–Jones Conjecture with finite wreath products (relative to $\mathcal{VCYC}$) argue as follows: G is the directed colimit of the G_n , where G_n was an HNN-extension of G_{n-1} over a finite edge group. Hence, G_n is hyperbolic relative to G_{n-1} . Since G_0 is a directed colimit of finite groups it satisfies the Farrell–Jones Conjecture with finite wreath products relative to $\mathcal{VCYC}$ (again by [Weg15, Proposition 2.17]). Thus, by [Bar16, Corollary 4.6, Remark 4.7] and induction, all G_n satisfy the Farrell–Jones Conjecture with finite wreath products relative to $\mathcal{VCYC}$. All in all, by the above corollary, the group $G *__{G_0} G'$ satisfies the Farrell–Jones Conjecture with finite wreath products relative to $\mathcal{VCYC}$.

3.2 Acylindrical actions on trees and Waldhausen Nil-groups

The second main result of this thesis is concerned with Waldhausen Nil-groups $\widetilde{\operatorname{Nil}}_n(RC; R[A - C], R[B - C])$ associated to an amalgamated free product $G = A *_C B$ acting acylindrically on its Bass-Serre tree. For a precise, yet concise definition of the groups $\widetilde{\operatorname{Nil}}_n(RC; R[A - C], R[B - C])$ we refer the reader to [DQR11, Section 3].

Theorem 3.9. *Let $G = A *_C B$ act acylindrically on the associated Bass-Serre tree T . Let $\mathcal{L}$ be a set of representatives for the orbits of the action $G \curvearrowright \partial T \times \partial T \setminus \operatorname{diag}$. Then for any ring R there are isomorphisms*

$$\widetilde{\operatorname{Nil}}_{n-1}(RC; R[A - C], R[B - C]) \cong \bigoplus_{L \in \mathcal{L}} \operatorname{coker}(H_n^{GL}(E_{FLN} G_L; \mathbf{K}_R) \rightarrow H_n^{GL}(\operatorname{pt}; \mathbf{K}_R))$$

for $n \in \mathbb{Z}$.

In the special case where G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{VCYC}$ this was already established by Lafont and Ortiz [LO09].

The proof of Theorem 3.9 relies mainly on Theorem 3.1 and an identification result carefully carried out by Davis, Quinn and Reich in [DQR11]. Thus, for the special case where C is finite—in other words, for the case where $G = A *_C B$ is hyperbolic relative to $\{A, B\}$ —the above theorem could have already been obtained (following the same line of proof as we will) from the main result of [Bar16].

As a third remark on the above theorem, recall that the right hand has been identified with certain Farrell Nil-groups (see [DQR11, Lemma 3.1]) whose further properties were for instance be studied in [Gru07], [Ram07] and very recently, [LPW16].

Since all G_L appearing on the right hand side of the above isomorphism are virtually cyclic by Lemma 3.3, we can exploit what is known in general about the relative assembly map to obtain the following corollary. Recall that a ring R is called *regular*, if it is Noetherian and every finitely generated R -module has a finite dimensional projective resolution.

Corollary 3.10. *Let $G = A *_C B$ act acylindrically on its Bass-Serre tree T and let R be a regular ring. Then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ vanishes rationally. Furthermore,*

a) *if the action $G \curvearrowright T$ is strictly acylindrical (e.g. when C is malnormal in A or B), then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C]) = 0$;*

b) *if $\mathbb{Q} \subset R$, then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C]) = 0$.*

Proof. By [LS15, Theorem 0.3], the relative assembly map

$$H_n^{G_L}(E_{\mathcal{FIN}}G_L; \mathbf{K}_R) \rightarrow H_n^{G_L}(E_{\mathcal{VCYC}}G_L; \mathbf{K}_R)$$

is rationally bijective. Since all G_L are virtually cyclic, $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ vanishes rationally by Theorem 3.9.

a) If $G \curvearrowright T$ is strictly acylindrical, then any G_L is either trivial or $\mathbb{Z}$. Since R is assumed to be regular, the fundamental theorem of algebraic K -theory (Bass-Heller-Swan decomposition) [Ros94, 5.3.30 Theorem] implies that all $H_n^{G_L}(E_{\mathcal{FIN}}G_L; \mathbf{K}_R) \rightarrow H_n^{G_L}(\text{pt}; \mathbf{K}_R)$ are isomorphism.

b) By [KL05, Lemma 21.24], the relative assembly map for G_L is integrally an isomorphism. Since all G_L are virtually cyclic, the claim follows. $\square$

For the proof of Theorem 3.9 we need a suitable model for $E_{\mathcal{F}}G$, where $\mathcal{F} = \mathcal{F}_T \cup \mathcal{F}_{\partial}$ is again defined as in Notation 2.1. As with the analysis of $\mathcal{F}_{\partial}$ before the author expects the following construction to be well known, but could not find a reference in the literature.

Let $\mathcal{L}$ be a set of representatives for the orbits of the action $G \curvearrowright \partial T \times \partial T \setminus \text{diag}$. We think of $\mathcal{L}$ as a collection of bi-infinite geodesic rays L in T and we denote by G_L the stabiliser of the element in $\mathcal{L}$ given by L . In particular, each G_L is an element of $\mathcal{F}_{\partial}$. Denote by $\text{cone}(L)$ the simplicial complex (and hence CW-complex) obtained by taking the simplicial join $p_L * L$. Both L and $\text{cone}(L)$ are G_L -CW-complexes, and thus $G \times_{G_L} L$ and $G \times_{G_L} \text{cone}(L)$ are G -CW-complexes. Each $G \times_{G_L} L$, when viewed as a CW-complex, is the disjoint union of lines. The complex $G \times_{G_L} L$ contains one line for each element in the set $\{gL \mid g \in G\}$. Similarly, $G \times_{G_L} \text{cone}(L)$ viewed as a CW-complex is the disjoint union of coned off lines and contains one coned off line for each element in the set $\{\text{cone}(gL) \mid g \in G\}$.

Since T is a G -CW-complex as well, we can form the following pushout of G -CW-complexes, where i is the canonical inclusion induced by the inclusion of each line into its cone and j_L is the G -map given by sending $(g, t) \in G \times_{G_L} L$ to $gt \in gL \subset T$.

$$\begin{array}{ccc} \coprod_{L \in \mathcal{L}} G \times_{G_L} L & \xrightarrow{\coprod_{L \in \mathcal{L}} j_L} & T \\ \downarrow i & & \downarrow \\ \coprod_{L \in \mathcal{L}} G \times_{G_L} \text{cone}(L) & \longrightarrow & Y \end{array} \quad (3.1)$$

Lemma 3.11. *The G -CW-complex Y given by the above pushout is a model for $E_{\mathcal{F}}G$.*

Proof. The G -CW-complex Y is obtained from T by equivariantly gluing coned off lines onto T along their ‘baselines’. Thus, the intersection of (the images of) different cones in Y lies in T . In particular, if $L \neq L'$, then the cone point p_L over L is distinct from the cone point $p_{L'}$ over L' . It follows that each p_L is fixed exactly by the elements in G_L . Moreover, since Y is a G -CW-complex, elements of G that do not lie in some G_L can only fix points in T . Let $H \in \mathcal{F} = \mathcal{F}_T \cup \mathcal{F}_{\partial}$. If $H \notin \mathcal{F}_{\partial}$, then H fixes only points in T , and T^H is non-empty and contractible.

If $H \in \mathcal{F}_{\partial}$ and H contains an element of positive translation length (with respect to $G \curvearrowright T$), then H can not fix a point in T . Furthermore, such a group H fixes exactly one cone point p_L for a unique line L and—since it does not fix any point in T —it does not fix any other point in $\text{cone}(L)$. Hence, in this case $Y^H = p_L$.

If $H \in \mathcal{F}_{\partial}$ and all $h \in H$ have translation length zero, then any line L for which $Hp_L = p_L$ holds must be fixed pointwise by H . Hence, for any such line $\text{cone}(L)$ must be fixed pointwise by H as well. There might be more than one such line, but Y^H is still contractible: Since different cones can only intersect in T , we can first contract Y^H to T^H by simultaneously retracting all cones $\text{cone}(L)$ in Y^H to their respective lines L , which all lie in T^H . Then we contract T^H to a point.

Conversely, if for some $H \leq G$ the set Y^H is non-empty, then (since Y is a G -CW-complex) H must fix at least one 0-cell of Y . Thus, H either fixes at least one vertex of T or is a subgroup of at least one of the G_L . Then, by definition, H lies in $\mathcal{F}_T \cup \mathcal{F}_{\partial} = \mathcal{F}$. $\square$

It might be worth noting that the above construction of a model for $E_{\mathcal{F}_T \cup \mathcal{F}_{\partial}}G$ neither used that G is an amalgamated free product nor that the action of G on its Bass-Serre tree is acylindrical or any knowledge on the nature of elements in $\mathcal{F}_{\partial}$ for that matter. That is, Lemma 3.11 holds for arbitrary simplicial group actions on trees.

In the sequel we will use the fact that for every ring R there is an equivariant homology theory $H_*^?(-; \mathbf{K}_R)$ such that for all groups G the G -homology theory $H_*^G(-; \mathbf{K}_R)$ is the G -homology theory appearing in the formulation of the K -theoretic Farrell–Jones Conjecture with coefficients in the ring R (see [BEL08, Theorem 6.1] or Section A.2). In particular, we assume the reader is comfortable with the properties of an equivariant homology theory (see [Lüc02, Section 1] for a definition).

Proof of Theorem 3.9. Davis, Quinn and Reich, in [DQR11, Lemma 3.1], identified the group $\widetilde{\text{Nil}}_{n-1}(RC; R[A - C], R[B - C])$ with the cokernel of the map

$$\text{asmb}_n(G, \mathcal{F}_T, R) : H_n^G(E_{\mathcal{F}_T} G; \mathbf{K}_R) \rightarrow H_n^G(\text{pt}; \mathbf{K}_R).$$

By Theorem 3.1 a), this cokernel is isomorphic to the cokernel of the map

$$H_n^G(T; \mathbf{K}_R) = H_n^G(E_{\mathcal{F}_T} G; \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{F}} G; \mathbf{K}_R),$$

which can be determined by exploiting the properties of $H_*^?(-; \mathbf{K}_R)$ and the explicit model for $E_{\mathcal{F}} G$ constructed above. Applying the G -homology theory $H_*^G(-; \mathbf{K}_R)$ (abbreviated to $H_*^G(-)$ in the following) to the pushout (3.1) yields a Mayer-Vietoris sequence

$$\dots \rightarrow \bigoplus_{L \in \mathcal{L}} H_n^G(G \times_{G_L} L) \rightarrow H_n^G(T) \oplus \bigoplus_{L \in \mathcal{L}} H_n^G(G \times_{G_L} \text{cone}(L)) \rightarrow H_n^G(E_{\mathcal{F}} G) \rightarrow \dots$$

The space $\text{cone}(L)$ is G_L -equivariantly contractible and L is a model for $E_{\mathcal{F} \cap \mathcal{N}} G_L$. Thus, using the induction structure of $H_*^?$ for the G -CW-complexes $G \times_{G_L} L = \text{ind}_{G_L}^G L$ and $G \times_{G_L} \text{cone}(L) = \text{ind}_{G_L}^G(\text{cone}(L))$, the above sequence becomes

$$\dots \rightarrow \bigoplus_{L \in \mathcal{L}} H_n^{G_L}(E_{\mathcal{F} \cap \mathcal{N}} G_L) \xrightarrow{(f_1, f_2)} H_n^G(T) \oplus \bigoplus_{L \in \mathcal{L}} H_n^{G_L}(\text{pt}) \rightarrow H_n^G(E_{\mathcal{F}} G) \rightarrow \dots \quad (3.2)$$

Furthermore, the map f_2 is induced by the projections $\text{pr}_L : E_{\mathcal{F} \cap \mathcal{N}} G_L \rightarrow \text{pt}$. In other words, $f_2 = \bigoplus_{L \in \mathcal{L}} \text{asmb}_n(G_L, \mathcal{F} \cap \mathcal{N}, R)$. Since the relative assembly map

$$H_*^{G'}(E_{\mathcal{F} \cap \mathcal{N}} H) \rightarrow H_*^{G'}(E_{\mathcal{V} \cap \mathcal{C} \cap \mathcal{N}} H)$$

is split injective for all groups G' and rings R [Bar03, Theorem 1.3] and all G_L are virtually cyclic by Lemma 3.3, the maps $\text{asmb}_n(G_L, \mathcal{F} \cap \mathcal{N}, R)$ are split injective. Hence, f_2 is split injective as well and the long exact sequence (3.2) gives rise to a short exact sequence

$$0 \rightarrow \bigoplus_{L \in \mathcal{L}} \mathcal{H}_n^{G_L}(E_{\mathcal{F} \cap \mathcal{N}} G_L) \xrightarrow{(f_1, f_2)} \mathcal{H}_n^G(T) \oplus \bigoplus_{L \in \mathcal{L}} \mathcal{H}_n^{G_L}(\text{pt}) \rightarrow \mathcal{H}_n^G(E_{\mathcal{F}} G) \rightarrow 0$$

for every $n \in \mathbb{Z}$. By basic yoga with short exact sequences one can obtain from any short exact sequence of abelian groups

$$0 \rightarrow V_0 \xrightarrow{(i, j)} V_2 \oplus V_1 \rightarrow V_3 \rightarrow 0$$

with j injective, a short exact sequence of the form $0 \rightarrow V_2 \rightarrow V_3 \rightarrow \text{coker}(i) \rightarrow 0$. Thus, we obtain for all $n \in \mathbb{Z}$ the short exact sequence

$$0 \rightarrow H_n^G(T; \mathbf{K}_R) \xrightarrow{i_*} H_n^G(E_{\mathcal{F}} G; \mathbf{K}_R) \rightarrow \text{coker} \bigoplus_{L \in \mathcal{L}} \text{asmb}_n(G_L, \mathcal{F} \cap \mathcal{N}, R) \rightarrow 0$$

from which the claim follows. $\square$

In light of Corollary 3.10 a) and Waldhausen's notion of regular coherent groups (see [Wal78b, Section 19]) we conclude with the following remark.

Remark 3.12. For all regular coherent groups C and all regular rings R , the group $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ already vanishes integrally by [Wal78b, Theorem 11.2]. However, it is easy to find examples of amalgamated free products $G = A *_C B$ acting strictly acylindrically on their Bass-Serre tree, such that C is not regular coherent, because it contains torsion. For instance, take $C = \mathbb{Z}/2\mathbb{Z} \times F_2$ malnormal in $A := D_\infty \times A'$ for any torsion-free hyperbolic group A' and choose any fitting group B .

Appendix A

The Farrell-Jones Conjecture and directed colimits

The purpose of this appendix is to give the proof omitted in Chapter 1 of the following lemma.

Lemma 1.6. *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). Let $G = \operatorname{colim}_{i \in I} G_i$, with structure maps $\phi_i : G_i \rightarrow G$, and let $\mathcal{F}$ be a family of subgroups of G . If for all $i \in I$ the group G_i satisfies the K -theoretic (resp. L -theoretic) Farrell-Jones Conjecture relative to $\phi_i^* \mathcal{F} := \{H \leq G_i \mid \phi_i(H) \in \mathcal{F}\}$, then G satisfies the K -theoretic (resp. L -theoretic) Farrell-Jones Conjecture relative to $\mathcal{F}$.*

In the L -theoretic case, if $\mathcal{F} = \mathcal{VCYC} = \mathcal{VCYC}(G)$ and $\phi_i^* \mathcal{F}$ is replaced by $\mathcal{VCYC}(G_i)$ this lemma is [BL10, Corollary 0.8]. We wish to emulate the proof of [BL10, Corollary 0.8] for the setting of general families, while at the same time we want to avoid using any statements that require the families involved to come from a class of groups that is closed under isomorphisms and taking quotients.

The idea given in [BL10] is to derive [BL10, Corollary 0.8] from suitable results in [BEL08, Section 5]. This requires the existence of strongly continuous equivariant homology theories (over a fixed group Γ) that give the G -homology theories appearing in the formulation of the Farrell-Jones conjecture with coefficients in additive G -categories (with involution). The existence of these theories is common knowledge among the ‘Farrell-Jones community’. However, there seems to be no reference stating their existence in the generality needed for our purposes. Therefore, in Section A.1 we will give a proof of Lemma 1.6 using the existence of the required equivariant homology theories. For completeness we elaborate on the construction of these theories afterwards, in Section A.2.

Throughout the whole appendix we focus on the L -theoretic case, that is—aside from remarks—everything is formulated for additive G -categories with involution.

A.1 The proof of Lemma 1.6 (modulo the existence of suitable equivariant homology theories)

We will obtain Lemma 1.6 as a corollary from [BEL08, Theorem 5.2] and a straightforward generalisation of [BL10, Theorem 0.7]. For this we first have to introduce the fibred version of the Farrell-Jones Conjecture with coefficients in additive G -categories.

Notation A.1. Let G be a group, $\mathcal{F}$ be a family of subgroups of G and $\mathcal{A}$ be an additive G -category with involution. We say that $\text{LFJC}(G, \mathcal{F}, \mathcal{A})$ holds, if G satisfies the *L-theoretic Farrell-Jones Conjecture relative to $\mathcal{F}$ with coefficients in $\mathcal{A}$* , i.e. if the assembly map

$$\text{asmb}_n(G, \mathcal{F}, \mathcal{A}) : H_n^G(E_{\mathcal{F}}G; \mathbf{L}_{\mathcal{A}}^{-\infty}) \rightarrow H_n^G(\text{pt}; \mathbf{L}_{\mathcal{A}}^{-\infty})$$

is an isomorphism for all $n \in \mathbb{Z}$. We say that $\text{LFJC}(G, \mathcal{F})$ holds if $\text{LFJC}(G, \mathcal{F}, \mathcal{A})$ holds for all additive G -categories with involution $\mathcal{A}$. In other words, $\text{LFJC}(G, \mathcal{F})$ holds if G satisfies the *L-theoretic Farrell-Jones Conjecture relative to $\mathcal{F}$* (as in Definition 1.2).

Definition A.2. (cf. [BL10, Definition 0.6]) A group G satisfies the *fibred L-theoretic Farrell-Jones Conjecture relative to $\mathcal{F}$* (in short: $\text{fibLFJC}(G, \mathcal{F})$ holds) if for any group homomorphism $\mu : G' \rightarrow G$ and any additive G -category with involution $\mathcal{A}$ the statement $\text{LFJC}(G', \mu^*\mathcal{F}, \mu^*\mathcal{A})$ holds. Here, $\mu^*\mathcal{F} = \{H \leq G' \mid \mu(H) \in \mathcal{F}\}$ and $\mu^*\mathcal{A}$ is given by letting G' act on $\mathcal{A}$ via restriction along μ .

Theorem A.3. (cf. [BL10, Theorem 0.7]) Let G be a group. Then $\text{LFJC}(G, \mathcal{F})$ holds if and only if $\text{fibLFJC}(G, \mathcal{F})$ holds.

Proof. For $\mathcal{F} = \mathcal{VCYC}$ this is [BL10, Theorem 0.7]. Furthermore, the only statement in the proof of [BL10, Theorem 0.7] involving the family of subgroups is the fact that

$$\mu^*E_{\mathcal{VCYC}}G \simeq E_{\mu^*\mathcal{VCYC}}G'$$

holds. Since this is still true if one replaces $\mathcal{VCYC}$ by $\mathcal{F}$, the claim follows. $\square$

Proof of Lemma 1.6. In the *L-theoretic* case the assertion of the lemma is

$$\forall i \in I : \text{LFJC}(G_i, \phi_i^*\mathcal{F}) \Rightarrow \text{LFJC}(G, \mathcal{F}).$$

By Theorem A.3, this is equivalent to

$$\forall i \in I : \text{fibLFJC}(G_i, \phi_i^*\mathcal{F}) \Rightarrow \text{LFJC}(G, \mathcal{F}).$$

For any additive G -category with involution $\mathcal{A}$ there is a strongly continuous equivariant homology theory $\mathcal{H}_*^?$ over G such that for any homomorphism $\mu : G' \rightarrow G$ the G' -homology theory $\mathcal{H}_*^{(G', \mu)}$ is isomorphic to the G' -homology theory $H_*^{G'}(-; \mathbf{L}_{\mu^*\mathcal{A}}^{-\infty})$ appearing in the formulation of the *L-theoretic Farrell-Jones Conjecture for G'* (see Lemma A.4). Hence, for each additive G -category with involution $\mathcal{A}$ we can apply [BEL08, Theorem 5.2]. So, all in all, we obtain that $\text{fibLFJC}(G, \mathcal{F})$ and, in particular, $\text{LFJC}(G, \mathcal{F})$ holds. The *K-theoretic* case works analogously. $\square$

A.2 The construction of suitable equivariant homology theories

The purpose of this section is to elaborate on the following well-known statement and its (also well-known) K -theoretic analogue.

Lemma A.4. *Let G be a group and $\mathcal{A}$ be an additive G -category with involution. Then there is a strongly continuous equivariant homology theory $\mathcal{H}_*^?$ over G such that for all group homomorphisms $\mu : G' \rightarrow G$ the G' -homology theory $\mathcal{H}_*^{(G', \mu)}$ is isomorphic to the G' -homology theory $H_*^{G'}(-; \mathbf{L}_{\mu^* \mathcal{A}}^{-\infty})$ appearing in the formulation of the L -theoretic Farrell-Jones Conjecture for G' .*

All the crucial pieces needed to obtain this lemma can be found in [BEL08] and [BL10]. Thus, writing down a proof for Lemma A.4 amounts to connecting these pieces in the right order. We assume throughout this section, that the reader is comfortable with or willing to look up the relevant sections of [DL98], [BEL08] and [BL10]. In particular, the definition of an equivariant homology theory over a fixed group Γ (see [BEL08, Definition 2.3]) will not be explained.

We start by fixing some notation. Denote by $\mathbf{Spectra}$ the category of spectra with strict maps of spectra (see [BEL08, Section 7]). Furthermore, let $\mathbf{AddCat}_{\text{inv}}$ be the category of small additive categories with involution as in [BL10, the beginning of Section 7 resp. Definition 1.4]. For any group Γ we denote by $\mathbf{Groupoids} \downarrow \Gamma$ the category of small (not necessarily connected) groupoids over Γ (cf. [BEL08, Section 7, before Lemma 7.1]) and by $\mathbf{connGroupoids} \downarrow \Gamma$ the category of connected groupoids over Γ . In particular, we have the notion of a group over Γ .

A.2.1 G -homology theories for the Farrell-Jones Conjecture

In this section we review the construction of the G -homologies theories appearing in the formulation of the Farrell-Jones Conjecture.

Let $\mathbb{E} : \mathbf{AddCat}_{\text{inv}} \rightarrow \mathbf{Spectra}$ be a covariant functor which sends weak equivalences of additive categories with involution to weak equivalences of spectra.

Let $\mathcal{A}$ be an additive G -category with involution. In particular, there is a right G -action on $\mathcal{A}$ (by functors of additive categories). Then, if $F : \mathcal{G} \rightarrow G$ is a connected groupoid over G , one can form the contravariant functor

$$F_{\mathcal{A}} : \mathcal{G} \rightarrow G \rightarrow \mathbf{AddCat}_{\text{inv}},$$

which sends each object of $\mathcal{G}$ to $\mathcal{A}$ and each morphism f in $\mathcal{G}$ to the endomorphism of $\mathcal{A}$ given by $F(f)$.

For any contravariant functor $F : \mathcal{G} \rightarrow \mathbf{AddCat}_{\text{inv}}$, where $\mathcal{G}$ is a connected groupoid, the Grothendieck type construction in [BL10, Section 7, (7.2)] yields an additive category

with involution $\int_{\mathcal{G}} F$ (where we suppress the involution in the notation). By [BL10, Lemma 7.11], there is a functor

$$\begin{aligned} \int_{\mathcal{G}}(-)_{\mathcal{A}} : \mathbf{connGroupoids} \downarrow G &\rightarrow \mathbf{AddCat}_{\text{inv}} \\ (F : \mathcal{G} \rightarrow G) &\mapsto \int_{\mathcal{G}} F_{\mathcal{A}}, \end{aligned}$$

which sends an equivalence of connected groupoids over G to an equivalence of additive categories with involution. Denote by $\mathbf{E}'_{\mathcal{A}}$ the composition $\mathbf{E}'_{\mathcal{A}} := \mathbb{E} \circ \int_{\mathcal{G}}(-)_{\mathcal{A}}$.

To obtain an $\text{Or}(G)$ -spectrum to which one can associate a G -homology theory via the Davis-Lück construction, a transport functor $t^G : \text{Or}(G) \rightarrow \mathbf{connGroupoids} \downarrow G$ is constructed as follows: For any group G and any homogeneous space G/H denote by $t^G(G/H)$ the groupoid whose set of objects is G/H and whose set of morphisms from g_1H to g_2H is given by the set $\{g \in G \mid gg_1 = g_2\}$. The functor t^G is then given by $G/H \mapsto (t^G(G/H) \rightarrow t^G(G/G) = G)$. Finally, denote by $\mathbf{E}_{\mathcal{A}} := \mathbf{E}'_{\mathcal{A}} \circ t^G$ the composition

$$\mathbf{E}_{\mathcal{A}} : \text{Or}(G) \xrightarrow{t^G} \mathbf{connGroupoids} \downarrow G \xrightarrow{\int_{\mathcal{G}}(-)_{\mathcal{A}}} \mathbf{AddCat}_{\text{inv}} \xrightarrow{\mathbb{E}} \mathbf{Spectra}$$

and by $H_*^G(-; \mathbf{E}_{\mathcal{A}})$ the G -homology theory obtained from $\mathbf{E}_{\mathcal{A}}$ by the construction in [DL98, Section 4].

Remark A.5. If $\mathbb{E} = \mathbb{L}^{-\infty}$ is as in [CP95, Definition 4.16] (i.e. $\mathbb{L}^{-\infty}$ is the functor sending an additive category with involution to its algebraic $L^{-\infty}$ -spectrum following Ranicki), then for any additive G -category with involution $\mathcal{A}$ the above construction yields the G -homology theory $H_*^G(-; \mathbf{L}_{\mathcal{A}}^{-\infty})$ appearing in the formulation of the L -theoretic Farrell-Jones Conjecture for G . The analogous statement is true for the functor $\mathbb{E} = \mathbb{K}^{-\infty}$ sending an additive category to its non-connective algebraic K -theory spectrum (see for instance [BFJR04, Sections 2.1 and 2.5]) and the K -theoretic Farrell-Jones Conjecture (cf. [BL10, Remark 10.8]).

A.2.2 Equivariant homology theories over a fixed group

This section describes how one can obtain an equivariant homology theory over a fixed group starting, again, with a covariant functor $\mathbb{E} : \mathbf{AddCat}_{\text{inv}} \rightarrow \mathbf{Spectra}$ sending weak equivalences of additive categories with involution to weak equivalences of spectra.

Let Γ be a fixed group and $\mathcal{A}$ an additive Γ -category with involution. Then, if $F : \mathcal{G} \rightarrow \Gamma$ is a (not necessarily connected) groupoid over Γ , one can, again, form the contravariant functor

$$F_{\mathcal{A}} : \mathcal{G} \rightarrow \Gamma \rightarrow \mathbf{AddCat}_{\text{inv}},$$

which sends each object of $\mathcal{G}$ to $\mathcal{A}$ and each morphism f in $\mathcal{G}$ to the endomorphism of $\mathcal{A}$ given by $F(f)$.

The construction of [BL10, Section 7] was extended in [BL10, Section 11] to yield for any contravariant functor $F : \mathcal{G} \rightarrow \mathbf{AddCat}_{\text{inv}}$, where $\mathcal{G}$ is a any groupoid, a $\mathbb{Z}$ -category with involution $\int_{\mathcal{G}} F$. Denote by $\mathbb{Z}\text{-Cat}_{\text{inv}}$ the category of $\mathbb{Z}$ -categories with involution as in [BL10, Definition 10.1]. By [BL10, p. 48, after Theorem 11.3], there is, again, a functor

$$\begin{aligned} \int_{\Gamma} (-)_{\mathcal{A}} : \mathbf{Groupoids} \downarrow \Gamma &\rightarrow \mathbb{Z}\text{-Cat}_{\text{inv}} \\ F : \mathcal{G} \rightarrow \Gamma &\mapsto \int_{\mathcal{G}} F_{\mathcal{A}}, \end{aligned}$$

which sends an equivalence of groupoids over Γ to an equivalence of $\mathbb{Z}$ -categories with involution. Moreover, in [BL10, Section 10] an adjoint pair of functors

$$\oplus : \mathbb{Z}\text{-Cat}_{\text{inv}} \rightarrow \mathbf{AddCat}_{\text{inv}, \oplus}, \quad \text{forget} : \mathbf{AddCat}_{\text{inv}, \oplus} \rightarrow \mathbb{Z}\text{-Cat}_{\text{inv}}$$

was constructed, where $\mathbf{AddCat}_{\text{inv}, \oplus}$ is the category of additive categories with functorial direct sum and involution (see [BL10, Definition 10.3] for the definition). So we can denote by $\mathcal{E}_{\mathcal{A}}$ the following composition:

$$\begin{aligned} \mathcal{E}_{\mathcal{A}} : \mathbf{Groupoids} \downarrow \Gamma &\xrightarrow{\int_{\Gamma} (-)_{\mathcal{A}}} \mathbb{Z}\text{-Cat}_{\text{inv}} \xrightarrow{(-)_{\oplus}} \mathbf{AddCat}_{\text{inv}} \xrightarrow{\mathbb{E}} \mathbf{Spectra} \\ (F : \mathcal{G} \rightarrow \Gamma) &\mapsto \mathbb{E}\left(\left(\int_{\mathcal{G}} F_{\mathcal{A}}\right)_{\oplus}\right) \end{aligned}$$

which again sends equivalences of groupoids over Γ to weak equivalences of spectra.

The definition of the transport functor can also easily be extended to groups over Γ . For any group (G', μ) over Γ define $t^{(G', \mu)}$ as

$$\begin{aligned} t^{(G', \mu)} : \text{Or}(G') &\rightarrow \mathbf{connGroupoids} \downarrow \Gamma, \\ G'/H &\mapsto (t^{G'}(G'/H) \rightarrow t^{G'}(G'/G') = G' \xrightarrow{\mu} \Gamma). \end{aligned}$$

Therefore, we have for any group (G', μ) over Γ an $\text{Or}(G')$ -spectrum

$$\mathcal{E}_{\mathcal{A}} \circ t^{(G', \mu)} : \text{Or}(G') \rightarrow \mathbf{Spectra},$$

to which we can associate a G' -homology theory by [DL98, Section 4]. Moreover, the following lemma tells us that these G' -homology theories fit nicely together and give an equivariant homology theory (that is we obtain one equivariant homology theory for each fixed Γ and $\mathcal{A}$).

Lemma A.6. (*[BEL08, Lemma 7.1 and its proof]*) *Let Γ be a fixed group and let $\mathcal{E} : \mathbf{Groupoids} \downarrow \Gamma \rightarrow \mathbf{Spectra}$ be a covariant functor which sends equivalences of groupoids to weak equivalences of spectra. Then there is an equivariant homology theory $\mathcal{H}_*^?(-; \mathcal{E})$ (with values in $\mathbb{Z}$ -modules) over Γ . In fact, for any group (G', μ) over Γ the G' -homology theory $\mathcal{H}_*^{(G', \mu)}(-; \mathcal{E})$ is the G' -homology theory obtained from the $\text{Or}(G')$ -spectrum $\mathcal{E} \circ t^{(G', \mu)}$ by the Davis-Lück construction.*

A.2.3 Comparing both constructions

Let $\mu : G' \rightarrow G$ be any group homomorphism and $\mathcal{A}$ be an additive G -category with involution. Denote by $\mu^*\mathcal{A}$ the additive G' -category with involution obtained by precomposing the action of G on $\mathcal{A}$ with μ .

Lemma A.7. *For any group homomorphism $\mu : G' \rightarrow G$ and any additive G -category with involution $\mathcal{A}$, the G' -homology theories $H_*^{G'}(-; \mathbf{E}_{\mu^*\mathcal{A}})$ and $\mathcal{H}_*^{(G', \mu)}(-; \mathcal{E}_{\mathcal{A}})$ are isomorphic, i.e. there is a natural transformation τ of G' -homology theories such that $\tau(X, A)$ is an isomorphism for all G' -CW-pairs (X, A) .*

Proof. Both $H_*^{G'}(-; \mathbf{E}_{\mu^*\mathcal{A}})$ and $\mathcal{H}_*^{(G', \mu)}(-; \mathcal{E}_{\mathcal{A}})$ are obtained by using the Davis-Lück construction on the $\text{Or}(G')$ -spectra $\mathbf{E}_{\mu^*\mathcal{A}} = \mathbf{E}'_{\mu^*\mathcal{A}} \circ t^{G'}$ and $\mathcal{E}_{\mathcal{A}} \circ t^{(G', \mu)}$ respectively. Hence, by [DL98, Lemma 4.6] it suffices to show there is a map f of $\text{Or}(G')$ -spectra between these two such that $f(G'/H)$ is a weak equivalence of spectra for all $G'/H \in \text{Obj}(\text{Or}(G'))$.

The two $\text{Or}(G')$ -spectra in question fit into the following diagram of functors and categories:

$$\begin{array}{ccccc}
 & \text{connGroupoids} \downarrow G' & \xrightarrow{f_?(-)_{\mu^*\mathcal{A}}} & \mathfrak{AddCat}_{\text{inv}} & \xrightarrow{\mathbb{E}} & \mathfrak{Spectra} \\
 \text{Or}(G') & \nearrow t^{G'} & \downarrow \mu_* & \nearrow f_?(-)_{\mathcal{A}} & & \\
 & \text{connGroupoids} \downarrow G & \xrightarrow{f_?(-)_{\mathcal{A}}} & \mathfrak{AddCat}_{\text{inv}} \subset \mathbb{Z}\text{-Cat}_{\text{inv}} & & \\
 & \nwarrow t^{(G', \mu)} & \nearrow f_?(-)_{\mathcal{A}} & \nwarrow (-)_{\oplus} & &
 \end{array}
 \quad (1) \quad (2)$$

with $\mathbf{E}_{\mu^*\mathcal{A}}$ being the topmost composition and $\mathcal{E}_{\mathcal{A}} \circ t^{(G', \mu)}$ the bottom most.

(1): This triangle commutes on the nose, since for a covariant functor $F : \mathcal{G} \rightarrow G'$ the contravariant functor $(\mu \circ F)_{\mathcal{A}} : \mathcal{G} \rightarrow G' \rightarrow G \rightarrow \mathfrak{AddCat}_{\text{inv}}$ is the same as the functor $F_{\mu^*\mathcal{A}} : \mathcal{G} \rightarrow G' \rightarrow \mathfrak{AddCat}_{\text{inv}}$.

(2): This triangle does not commute on the nose, but by [BL10, Lemma 10.4 (ii)] there is a natural isomorphism between the two functors

$$\left(\int_?(-)_{\mathcal{A}} \right)_{\oplus}, \int_?(-)_{\mathcal{A}} : \text{connGroupoids} \downarrow G \rightarrow \mathfrak{AddCat}_{\text{inv}}.$$

Since, by assumption, $\mathbb{E}$ sends equivalences of additive categories with involution to weak equivalences of spectra, the above diagram gives the desired map between the two $\text{Or}(G')$ -spectra $\mathbf{E}_{\mu^*\mathcal{A}} = \mathbf{E}'_{\mu^*\mathcal{A}} \circ t^{G'}$ and $\mathcal{E}_{\mathcal{A}} \circ t^{(G', \mu)}$. $\square$

A.2.4 Strongly continuous equivariant homology theories over a fixed group

The equivariant homology theories $\mathcal{H}_*^?(-; \mathcal{E}_{\mathcal{A}})$ turn out to be strongly continuous in the following sense.

Definition A.8. ([BEL08, Definition 3.3]) An equivariant homology theory $\mathcal{H}_*^?$ over a group Γ is called *strongly continuous* if for every group (G, ξ) over Γ and every directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i : G_i \rightarrow G$ for $i \in I$ the map

$$\operatorname{colim}_{i \in I} a_n(\psi_i) : \operatorname{colim}_{i \in I} \mathcal{H}_n^{(G_i, \xi \circ \psi_i)}(\operatorname{pt}) \rightarrow \mathcal{H}_n^{(G, \xi)}(\operatorname{pt})$$

is an isomorphism for all $n \in \mathbb{Z}$, where

$$a_n(\psi_i) : \mathcal{H}_n^{(G_i, \xi \circ \psi_i)}(\operatorname{pt}) \xrightarrow{\operatorname{ind}_{\psi_i}} \mathcal{H}_n^{(G, \xi)}(\operatorname{ind}_{\psi_i} \operatorname{pt}) \xrightarrow{\mathcal{H}_n^{(G, \xi)}(pr)} \mathcal{H}_n^{(G, \xi)}(\operatorname{pt}).$$

Lemma A.9. *If $\mathbb{E} : \mathfrak{AddCat}_{\operatorname{inv}} \rightarrow \mathfrak{Spectra}$ is compatible with colimits over directed systems (i.e. if $\mathcal{C} = \operatorname{colim}_{i \in I} \mathcal{C}_i$, then the canonical map $\operatorname{colim}_{i \in I} \mathbb{E}(\mathcal{C}_i) \rightarrow \mathbb{E}(\mathcal{C})$ is a weak equivalence of spectra), then for all additive Γ -categories with involution $\mathcal{A}$ the equivariant homology theory $\mathcal{H}_*^?(-; \mathcal{E}_{\mathcal{A}})$ over Γ is strongly continuous.*

Proof. Let (G, ξ) be a group over Γ , $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i : G_i \rightarrow G$ (that do not have to be injective). Furthermore, let $\phi_{ij} : G_i \rightarrow G_j$ be the maps constituting the directed system.

Unravelling the construction of the induction structure in [BEL08, proof of Lemma 7.1] one can see that the maps $a_n(\psi_i)$ in Definition A.8 can be identified with the maps induced by

$$\mathcal{E}_{\mathcal{A}}(\psi_i) : \mathcal{E}_{\mathcal{A}}(t^{G_i}(G_i/G_i) \downarrow \Gamma) \rightarrow \mathcal{E}_{\mathcal{A}}(t^G(G/G) \downarrow \Gamma)$$

on homotopy groups. It remains to convince ourselves that $\operatorname{colim}_{i \in I} \mathcal{E}_{\mathcal{A}}(\psi_i)$ behaves the way it should: Denote by $F_{\mathcal{A}} : t^G(G/G) \rightarrow \mathfrak{AddCat}_{\operatorname{inv}}$ and $(F_i)_{\mathcal{A}} : t^{G_i}(G_i/G_i) \rightarrow \mathfrak{AddCat}_{\operatorname{inv}}$ the functors associated to the groupoids $t^G(G/G) \downarrow \Gamma$ and $t^{G_i}(G_i/G_i) \downarrow \Gamma$, respectively (cf. Subsection A.2.2). For all $i \in I$, the functor of groupoids

$$t^{\psi_i} : t^{G_i}(G_i/G_i) \rightarrow t^G(G/\psi_i(G_i)) \rightarrow t^G(G/G)$$

sends a morphism $g_i \in G_i$ to the morphism $\psi_i(g_i) \in G$. Analogously for $i \leq j$ there are functors of groupoids

$$t^{\phi_{ij}} : t^{G_i}(G_i/G_i) \rightarrow t^{G_j}(G_j/\phi_{ij}(G_i)) \rightarrow t^{G_j}(G_j/G_j)$$

sending a morphism $g_i \in G_i$ to the morphism $\phi_{ij}(g_i) \in G_j$. Furthermore, those functors satisfy

$$(F_i)_{\mathcal{A}} = F_{\mathcal{A}} \circ t^{\psi_i} \quad \text{and} \quad (F_i)_{\mathcal{A}} = (F_j)_{\mathcal{A}} \circ t^{\phi_{ij}} \quad \text{for } i \leq j.$$

Hence, by [BL10, (7.7)] we obtain a directed system of additive categories with involution

$$\begin{array}{ccc} \int_{t^{G_i}(G_i/G_i)} (F_i)_{\mathcal{A}} & \xrightarrow{\quad} & \int_{t^{G_j}(G_j/G_j)} (F_j)_{\mathcal{A}} \\ & \searrow \quad \swarrow & \\ & \int_{t^G(G/G)} F_{\mathcal{A}} & \end{array}$$

That $\int_{t^G(G/G)} F_{\mathcal{A}}$ fulfils indeed the universal property of the colimit over this directed system can be seen directly by looking at the definition of the underlying additive categories (see [BL10, p. 22, (5.1)]): The crucial point is that morphisms in $\int_{t^G(G/G)} F_{\mathcal{A}}$ are finite formal sums $\sum_{g \in G} g\alpha_g$ with α_g being some morphism in $\mathcal{A}$. Therefore, any such morphism already appears as a morphism in some $\int_{t^{G_i}(G_i/G_i)} (F_i)_{\mathcal{A}}$ since the G_i were a directed system to begin with.

Since $\mathbb{E}$ was assumed to respect colimits over directed systems, it follows that the canonical map

$$\operatorname{colim}_{i \in I} \mathcal{E}_{\mathcal{A}}(t^{G_i}(G_i/G_i) \downarrow \Gamma) \rightarrow \mathcal{E}_{\mathcal{A}}(t^G(G/G) \downarrow \Gamma)$$

is indeed a weak equivalence of spectra. □

Remark A.10. Since the functors $\mathbb{K}$ and $\mathbb{L}^{-\infty}$ are compatible with colimits over directed systems (see [BL12a, Theorem 5.1 (iv)]), this concludes our elaboration on Lemma A.4.

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