Mathematik

On coset posets, nerve complexes and subgroup graphs of finitely generated groups

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Abstract

We study the homotopy type of the coset poset and the finite index coset poset of finitely generated infinite groups, focusing in particular on the contractibility. The coset poset $\mathscr{C}(G)$ of a group G is the set of all right cosets of all proper subgroups of G, ordered by inclusion. It was introduced by K.S. Brown. Little is known about the homotopy type of the order complex of the coset poset $\Delta\mathscr{C}(G)$. J. Shareshian and R. Woodroofe proved that the order complex $\Delta\mathscr{C}(G)$ is not contractible if G is a finite group. D.A. Ramras proved that the order complex $\Delta\mathscr{C}(G)$ is contractible for all infinite generated groups. Therefore we are interested in the case that G is a finitely generated infinite group. In fact, we also study the homotopy type of the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$, a subset of the coset poset. We prove that there exist examples of finitely generated infinite groups both for contractible and for non-contractible coset posets and finite index coset posets. Moreover, the order complexes $\Delta\mathscr{C}(G)$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are not necessarily homotopy equivalent for finitely generated infinite groups. Therefore we obtain two non-trivial different homotopy invariants. To prove this, we develop our theory of subgroup graphs of finite index subgroups of finitely generated groups and study the homotopy type of the nerve complexes $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$, which are homotopy equivalent to $\Delta\mathscr{C}(G)$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G)$, respectively.

In Erinnerung an M. Welsch
1995–2012

Wer im Gedächtnis seiner Lieben lebt, ist nicht tot. Er ist nur fern.
Tot ist nur, wer vergessen wird.

Immanuel Kant

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Contents

Notation

$\cong$	isomorphism	
$\simeq$	homotopy equivalence	
$G * H$	free product of the groups G and H	
$K * L$	join of the two simplicial complexes K and L	
L^0	0-skeleton of the simplicial complex L	
$G \times H$	direct product of the groups G, H	
G^n	direct product of n copies of G	
$N \rtimes_\psi H$	semidirect product of N with H and homomorphism $\psi\colon H \to \mathrm{Aut}(N)$	
$\mathbb{Z}_n$	cyclic group of order n	
$\mathbb{N}$	natural numbers including 0	
$\mathbb{N}_{>n}$	set of all natural numbers bigger than n	
$\mathbb{P}$	set of all primes	
$\Phi(G)$	Frattini subgroup of G, intersection of all maximal subgroups	
$\Phi_{\mathrm{fi}}(G)$	intersection of all maximal subgroups of finite index	(7.1.1)
$F(X)$	free group with finite generating set X	
χ	Euler characteristic	
$\tilde{\chi}$	reduced Euler characteristic $\tilde{\chi} = \chi - 1$	
ΔP	order complex of the poset P	Def. 2.2
$\mathcal{NC}(G, \mathscr{H})$	nerve complex of a group G with respect to a set of subgroups $\mathscr{H}$	Def. 2.9
$\bigcap \sigma := \bigcap_{\{Hg\} \in \sigma} Hg$	the intersection of cosets which are vertices	
$\mathscr{H}_{\mathscr{C}}(G)$	set of all proper subgroups of G	(2.0.1)
$\mathscr{C}(G)$	coset poset, set of all cosets of all proper subgroups in G	(2.0.2)
$\mathscr{H}_{\max}(G)$	set of all maximal subgroups of G	(2.0.3)
$\mathscr{H}_{\mathrm{fi}}(G)$	set of all proper finite index subgroups of G	(6.1.1)
$\mathscr{C}_{\mathrm{fi}}(G)$	finite index coset poset, set of all cosets of all proper finite index subgroups in G	(6.1.2)
$\mathscr{H}_{\max,\mathrm{fi}}(G)$	set of all maximal subgroups of finite index in G	(6.1.3)
$\mathscr{H}_{coH,\max,\mathrm{fi}}(G)$	set of all maximal finite index subgroups M with $HM = G$	(6.1.4)
$\mathscr{H}_{coH,\mathrm{fi}}(G)$	set of all proper finite index subgroups K with $HK = G$	(8.4.1)
$\mathscr{H}_{coH}(G)$	set of all proper subgroups K with $HK = G$	(8.4.2)
$\mathscr{H}_{\mathrm{nor},\max}(G)$	set of all normal maximal proper finite index subgroups	(6.1.6)
$\mathscr{H}_{\mathbb{P}}(G)$	all proper finite index subgroups with prime index	(6.1.8)
$\mathscr{H}_{\mathbb{P}^n}(G)$	all proper finite index subgroups with prime power index where the power is at most n	(6.1.10)
$\mathscr{H}_H(G)$	all proper subgroups K with $H \leq K$	(8.2.1)
$\mathscr{H}_{H,\mathrm{fi}}(G)$	all proper finite index subgroups K with $H \leq K$	(6.1.5)

1 Introduction

In this thesis we study the contractibility, connectivity, and homotopy type of simplicial complexes arising from finitely generated groups. The study of such simplicial complexes is located at the intersection of combinatorics, topology, and algebra. In particular, this thesis belongs to combinatorial group theory, since we use presentations of groups frequently.

The simplicial complex we are mainly interested in is the order complex of the coset poset. The coset poset $\mathscr{C}(G)$ is the set of all proper subgroups and their cosets, ordered by inclusion. Recall that we can apply topological concepts to a poset P by using the order complex ΔP whose simplices are the finite chains of P. We will sometimes use P to denote both ΔP and its geometric realization $|P| = |\Delta P|$.

The coset poset was introduced for finite groups by K.S. Brown in [Bro00]. There he considered the Euler characteristic χ of the coset poset, since it is connected to the probabilistic zeta function. The probabilistic zeta function $P(G, s)$ is the probability that a randomly chosen ordered s-tuple from a finite group G generates G. In [Bro00, Section 2] Brown showed how to compute an analytic continuation $P(G, -1)$ using a formula from P. Hall. In [Bro00, Section 3] he proved Bouc's observation that $P(G, -1) = -\tilde{\chi}(\mathscr{C}(G))$ with the reduced Euler characteristic $\tilde{\chi}(\mathscr{C}(G)) = \chi(\mathscr{C}(G)) - 1$ and $\chi(\mathscr{C}(G)) = \chi(\Delta\mathscr{C}(G))$. This result motivated Brown to study the homotopy type of the coset poset, which raised more questions than it answered (as Brown said). The question of simple connectivity was studied by D.A. Ramras in [Ram05]. Among other results he proved that the coset poset $\mathscr{C}(G)$ is contractible if G is an infinitely generated group. The question of contractibility was answered by J. Shareshian and R. Woodroofe in [SW16]. They proved that if G is a finite group, the coset poset $\mathscr{C}(G)$ is not contractible.

These results motivate us to study the homotopy type of the coset poset of a finitely generated infinite group, focusing on the contractibility. Furthermore, in some cases we even determine the connectivity or the homotopy type. In fact, we are not only interested in the coset poset, but also in an important subset, namely the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ of finitely generated groups. The finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ is the set of all proper finite index subgroups and their cosets, ordered by inclusion.

It is difficult to examine the homotopy type of these order complexes. Shareshian and Woodroofe used the classification of finite simple groups. Ramras used atomized posets and a simplicial complex with less vertices but higher dimension to prove his results. We use a simplicial complex which contains the order complex. It has the same vertex set but higher dimension. We call it nerve complex, which was studied by H. Abels and S. Holz in [AH93]. Basically, it is the nerve of the covering of a group by a set of subgroups and their cosets. In other literature, as in [BFM^{+}16], it is called the coset complex. The simplices of the nerve complexes $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ are the finite subsets of $\mathscr{C}(G)$ and $\mathscr{C}_{\mathrm{fi}}(G)$, respectively, with non-empty intersection. The set $\mathscr{H}_{\mathscr{C}}$ is defined to be the set of all proper subgroups of G and $\mathscr{H}_{\mathrm{fi}}$ is defined as the set of all proper finite index subgroups of G.

We prove that the order complexes $\Delta\mathscr{C}(G)$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are homotopy equivalent to the nerve complexes $\mathcal{NC}(G,\mathscr{H}_{\mathscr{C}})$ and $\mathcal{NC}(G,\mathscr{H}_{\mathrm{fi}})$, respectively. Since simplices are easier to recognize in the nerve complexes, we will consider them. Consequently, this thesis is also a further study of the connectivity of nerve complexes and of higher generation, as in the author's Master thesis [Wel].

At this point the nerve complexes give us a method to study the posets. Nevertheless, we need knowledge about the properties of the subgroups and their cosets. To handle at least the cosets of the finite index subgroups, we develop our theory of subgroup graphs of finite index subgroups. This is a very useful tool to study finite index subgroups of finitely generated groups, which was published in [Wel17].

We expect that both the finite index coset poset and the coset poset of a finitely generated group are contractible if and only if the set of all maximal subgroups of finite index is infinite. Although we are not able to prove this characterization, we do present various results in both directions. Furthermore, we expect that the coset poset is homotopy equivalent to a finite simplicial complex if and only if its set of all maximal subgroups is finite.

1.1 Main results

The main results are presented in this section as short versions of the original theorems.

1.1.1 Finite index coset poset

We start with the question of the contractibility of the finite index coset poset.

Corollary 1.1. (See Corollary 7.11)
There exist examples of finitely generated infinite groups both for contractible and for non-contractible finite index coset posets. Thus we obtain a non-trivial homotopy invariant.

We expect that the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ is contractible if and only if $\mathscr{H}_{\mathrm{max,fi}}(G)$ is infinite, where $\mathscr{H}_{\mathrm{max,fi}}(G)$ is the set of all maximal subgroups of finite index, see Conjecture 1.

We prove Corollary 1.1 in the following way. First, we consider the case that the finite index coset poset is contractible. Our main idea to prove this is the following theorem, in which $\mathscr{H}_{coH,\mathrm{max,fi}}(G)$ is defined for a subgroup $H \leq G$ as the set of all subgroups $M \in \mathscr{H}_{\mathrm{max,fi}}(G)$ such that $HM = G$.

Theorem 1.2. (Cone Argument, see Theorem 6.4)
Let G be a finitely generated infinite group. For every finite subcomplex U of the nerve complex $\mathcal{NC}(G,\mathscr{H}_{\mathrm{fi}})$, there exists a cone $U * \{K\}$ in the nerve complex if and only if $\mathscr{H}_{coH,\mathrm{max,fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\mathrm{fi}}(G)$. If the latter holds, the nerve complex $\mathcal{NC}(G,\mathscr{H}_{\mathrm{fi}})$ and therefore the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ are contractible.

The Cone Argument has to be combined with the following theorem.

Theorem 1.3. (See Theorems 6.6 and 6.9 and Corollary 6.12)
Let G be a finitely generated group. If G has infinitely many normal maximal subgroups, or infinitely many subgroups of prime index, or infinitely many subgroups of a bounded prime power index, then the set $\mathscr{H}_{coH,\max,\mathrm{fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\mathrm{fi}}(G)$.

Then we need to prove the existence of this infinite set of special subgroups. Therefore we develop the following theory of subgroup graphs of finite index subgroups of finitely generated groups. It is based on the subgroup graph theory of subgroups of free groups in [KM02] by Ilya Kapovich and Alexei Myasnikov.

Theorem 1.4. (Subgroup Graph, see Theorem 4.5)
Let G be a group with a presentation $G = \langle X \mid R \rangle$, where X is finite. Then there exists a one to one correspondence between the set of finite index subgroups of index n in G and the set of based, connected, X-regular graphs with n vertices (unique up to a canonical isomorphism of based X-graphs), which fulfill the defining relators R. The subgroup graph of H with respect to X and R will be denoted by $\Gamma_{X,R}(H)$.

The subgroup graph $\Gamma_{X,R}(H)$ of a finite index subgroup H is the Schreier coset graph of H with respect to X and R. Thus for a given finite index subgroup the subgroup graph is well-known. But the important part of this is the opposite direction of the equivalence. Given a presentation $G = \langle X \mid R \rangle$, to prove that there exists a subgroup of index n we only need to proof the existence of a subgroup graph with n vertices. Moreover, we can read off properties of a subgroup using its subgroup graph. For example one can determine for a finite index subgroup if it is normal, its normalizer, or the intersection of subgroups.

Theorem 1.3 combined with our subgroup graph theory provides various classes of finitely generated infinite groups whose finite index coset poset is contractible. Among the list of such groups are many important classes as free groups, free abelian groups, Artin groups, pure braid groups, right angled Coxeter groups, Baumslag-Solitar groups, Thompson's group F, many hyperbolic triangle groups, all euclidean triangle groups, Fuchsian groups of genus $g \geq 2$, HNN extensions, and infinite $\mathrm{SL}(n, \mathbb{Z})$. For more results see Section 6.2.

Furthermore, we study general cases which lead to the following main results. To prove them, we use our theory of subgroup graphs.

Corollary 1.5. (See Theorem 6.23 and Corollary 6.25)
Let $G_1, G_2, ..., G_n$ be finitely generated groups with $n \geq 2$ from which at least two have proper finite index subgroups. Then the nerve complex $\mathcal{NC}(G_1 * ... * G_n, \mathscr{H}_{\mathrm{fi}})$ and the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G_1 * ... * G_n)$ are contractible.

Corollary 1.6. (See Theorem 6.26 and Corollaries 6.27 and 6.28)
Let $G_1, ..., G_n$ be finitely generated groups. Suppose that G_1 contains infinitely many subgroups of prime power index. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ are contractible if G is either the free product $G_1 * ... * G_n$, the direct product $G_1 \times ... \times G_n$, or the semidirect product $G_n \rtimes (G_{n-1} \rtimes (... \rtimes (G_2 \rtimes G_1)...))$.

Theorem 1.7. (See Theorem 6.31)
Let G be a finitely generated group such that for each finite index subgroup $K < G$ the set $\mathscr{H}_{coK,\max,\mathrm{fi}}(G)$ is infinite. Let H be a finite index subgroup of G. Then the nerve complex $\mathcal{NC}(H, \mathscr{H}_{\mathrm{fi}})$ and the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(H)$ are contractible.

Lastly we study the case that the finite index coset poset is non-contractible. Certainly the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ is non-contractible if it is empty. Examples are the finitely generated infinite groups such as the Tarski monster groups and the Thompson's groups V and T. To obtain examples with non-empty finite index coset poset, we prove the finite index analogue of Brown's homotopy $\mathscr{C}(G) \simeq \mathscr{C}(G/N)$, where $\Phi_{\mathrm{fi}}(G)$ denotes the intersection of all maximal subgroups of finite index.

Theorem 1.8. (See Theorem 7.3)
Let G be a finitely generated group, N normal in G, and $N \subseteq \Phi_{\mathrm{fi}}(G)$. Then the simplicial complexes $\Delta\mathscr{C}_{\mathrm{fi}}(G)$, $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$, $\mathcal{NC}(G/N, \mathscr{H}_{\mathrm{fi}})$, and $\Delta\mathscr{C}_{\mathrm{fi}}(G/N)$ are homotopy equivalent.

This leads to the following result.

Corollary 1.9. (See Corollary 7.4)
Let G be a finitely generated group. Suppose that $\mathscr{H}_{\max,\mathrm{fi}}(G)$ is finite. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and therefore the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ are homotopy equivalent to the coset poset $\mathscr{C}(G/\Phi_{\mathrm{fi}}(G))$ and therefore are non-contractible and at most $(|\mathscr{H}_{\max,\mathrm{fi}}(G)| - 2)$-connected.

Examples of infinite groups whose finite index coset poset is non-contractible but non-empty are the Gupta-Sidki p-groups and the first Grigorchuk group. For more see Section 7.1.

1.1.2 Coset poset

After studying the finite index coset poset we turn to the coset poset.

Corollary 1.10. (See Corollary 8.13)
There exist examples of finitely generated infinite groups both for contractible and for non-contractible coset posets. Moreover, the coset poset and the finite index coset poset are not necessarily homotopy equivalent for finitely generated infinite groups. Thus we obtain an other non-trivial homotopy invariant.

Furthermore, we expect that the coset poset of a finitely generated infinite group is non-contractible if and only if $\mathscr{H}_{\max,\mathrm{fi}}$ is finite, see Conjecture 2.

First, we consider the contractible case.

Proposition 1.11. (Exchange Argument, see Proposition 8.2)
Let G be a finitely generated group such that $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ is contractible. Suppose that each infinite index subgroup K is contained in infinitely many finite index subgroups. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the coset poset $\mathscr{C}(G)$ are contractible.

The Exchange Argument provides the examples $\mathbb{Z} \rtimes F$ with F being a finite group. For more examples we need the following theorem, in which $\mathscr{H}_{H,\mathrm{fi}}(G)$ is defined for a subgroup H of G as the set of all finite index subgroups containing H .

Proposition 1.12. (Intersection Argument, see Proposition 8.4)
Let G be a finitely generated group such that the following hold.

(i) $\mathscr{H}_{H,\mathrm{fi}}(G)$ *or* $\mathscr{H}_{coH,\max,\mathrm{fi}}(G)$ *is infinite for each subgroup* $H \in \mathscr{H}_{\mathscr{C}}(G)$.

(ii) For every finite set of subgroups $\{H_1, ..., H_n\} \subseteq \mathscr{H}_{\mathscr{C}}(G)$ *the intersection* $\bigcap_{i=1}^{n} \mathscr{H}_{coH_i,\max,\mathrm{fi}}(G)$ *is infinite if* $\mathscr{H}_{H_i,\mathrm{fi}}(G)$ *is finite for* $i = 1, ..., n$.

Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ *and the coset poset* $\mathscr{C}(G)$ *are contractible.*

The Intersection Argument holds for all finitely generated groups which have infinitely many normal maximal subgroups. These are for example the free groups, free abelian groups, Artin groups, and pure braid groups. For more see Section 6.2.1.

Then we consider the non-contractible case. There already exist examples of finitely generated infinite groups whose coset poset is non-contractible. They are the Tarski monster groups and groups whose non-trivial proper subgroups are infinite cyclic. Ramras proved in [Ram05, Sections 3 and 6] that their coset poset is 0-connected but not 1-connected. We prove that the coset poset of $G \times (\mathbb{Z}_2)^n$ has the homotopy type of a wedge of infinitely many $(n+1)$-spheres if G is a Tarski monster group. Moreover, we prove that the coset poset of the first Grigorchuk group as well as the Gupta-Sidki p-groups are non-contractible and homotopy equivalent to their finite index coset posets. Thus their coset posets have the homotopy type of the coset poset of a finite group.

1.2 Structure of this thesis

This thesis is divided in three parts (plus an appendix). The first part (Section 2) introduces the simplicial complexes we are studying. After developing our theory of subgroup graphs in the second part (Sections 3–5) we are able to examine the finite index coset poset and the coset poset in the third part (Sections 6–8).

In Section 2 we set up notation and terminology. Moreover, we recall known results and add new results regarding the connectivity of the coset poset of finite groups.

Section 3 contains a brief summary of the relevant material on the theory of subgroup graphs of subgroups of free groups from Kapovich and Myasnikov [KM02]. Based on this we develop our theory of subgroup graphs of finite index subgroups of finitely generated groups in Section 4. In Section 5 some applications of our theory of subgroup graphs are indicated.

In Section 6 it is shown that the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ is contractible for various classes of finitely generated infinite groups. To prove this, we first show that the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are homotopy equivalent. Then we prove the Cone Argument, which is an important tool to show the contractibility in Section 6.1. Using the Cone Argument we prove that the finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is contractible if G contains infinite sets of special finite index subgroups. In Section 6.2 we use our theory of subgroup graphs to prove that these infinite sets of finite index subgroups exist for many important classes of groups. Section 6.3 exhibits some more general results about the contractibility of the finite index coset poset of finite products of finitely generated groups. Section 7 proves that there exist finitely generated infinite groups, apart from those where $\mathscr{C}_{\text{fi}}(G)$ is empty, for which the finite index coset poset is non-contractible. Therefore we prove that $\mathscr{C}_{\text{fi}}(G) \simeq \mathscr{C}_{\text{fi}}(G/N)$ for special N in Section 7.1. Moreover, in Section 7.2 we discuss the importance of the maximal subgroups for the homotopy type of the finite index coset poset. This leads to conclusions and new questions, which we state there. Finally, we study the coset poset $\mathscr{C}(G)$ of finitely generated infinite groups in Section 8. Analogously to the finite index coset poset we start with the contractible case in Section 8.1. There we prove the Exchange and Intersection Argument. Section 8.2 deals with the case that the coset poset is non-contractible. In Section 8.3 we study Tarski monster groups. This and the previous results lead to some questions and conclusions. In Section 8.4 we consider higher generation of a set of subgroups $\mathscr{H}$, which is defined using the connectivity of the nerve complex $\mathcal{NC}(G, \mathscr{H})$. With the results of this thesis we list the sets of subgroups and their higher generation.

In the appendix we list results which are connected to our theory of subgroup graphs and coset posets. In Appendix A we study posets related to the finite index coset poset. Appendix B gives examples of the usage of our subgroup graph theory. In Appendix C we prove that the finite index coset poset is contractible for many infinite triangle groups.

Part I
Finite simplicial complexes

The aim of this thesis is to study the homotopy type of the order complex of the coset poset of a finitely generated infinite group, focusing in particular on the contractibility. We examine the order complex using a corresponding nerve complex. Since the groups we consider are infinite, these complexes are infinite and therefore harder to picture. Thus we start this thesis considering finite groups. Moreover, we briefly sketch the so far known properties of the homotopy type of the coset poset.

2 The coset poset of finite groups

This section is mainly an introduction to coset posets and nerve complexes. First, we define the terms coset poset $\mathscr{C}(G)$ and order complex. Then we recall important results on the homotopy type of the coset poset, which are known so far. Afterwards we introduce the nerve complex in Definition 2.9. We prove in Corollary 2.11 that for all groups G the order complex $\Delta\mathscr{C}(G)$ and the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ are homotopy equivalent. Therefore we can study the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ to obtain results for the order complex $\Delta\mathscr{C}(G)$. In the last part of this section we prove the following results. For a finite group G the order complex $\Delta\mathscr{C}(G)$ is homotopy equivalent to the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\max})$, where $\mathscr{H}_{\max}$ is the set of all maximal subgroups of G, see Corollary 2.14. This leads to Corollary 2.19, which states that the coset poset $\mathscr{C}(G)$ of a finite group G is at most $(|\mathscr{H}_{\max}(G)| - 2)$-connected. Finally, we prove that for every $n \in \mathbb{N}$, there exists a finite group G_n such that $\mathscr{C}(G_n)$ is n-connected but not $(n+1)$-connected, see Proposition 2.20.

K.S. Brown introduced the coset poset in his influential paper [Bro00]. There he defined the coset poset as the poset of all proper left cosets of a finite group. In this thesis we use right cosets, which makes no difference, since $gH = (gHg^{-1})g$. We define the coset poset of a general group G as follows.

Definition 2.1. (Coset poset $\mathscr{C}(G)$)
Let G be a group and let

$$\mathscr{H}_{\mathscr{C}}(G) := \{H \mid H \text{ proper subgroup of } G\}. \tag{2.0.1}$$

The *coset poset* $\mathscr{C}(G)$ is the set

$$\mathscr{C}(G) := \{Hg \mid g \in G, H \in \mathscr{H}_{\mathscr{C}}(G)\}, \tag{2.0.2}$$

ordered by inclusion.

If G is understood, we write $\mathscr{H}_{\mathscr{C}}$ instead of $\mathscr{H}_{\mathscr{C}}(G)$. To study the probabilistic zeta function $P(G, s)$ Brown studied the Euler characteristic of the coset poset $\mathscr{C}(G)$. To any poset (partially ordered set) P one can apply topological concepts using the order complex ΔP.

Definition 2.2. (Order Complex, see [Bjö95, Section 9.3])
The *order complex* ΔP of a poset $P = (P, \subseteq)$ is the simplicial complex whose n-simplices are the chains $C_0 \subset ... \subset C_n$ of size $n + 1$ with $C_i \in P$.

Whenever we make topological statements about P or ΔP we have the geometric realization $|P| := |\Delta P|$ in mind.

Recall that a space X is n-*connected* for $n \geq 1$ if and only if X is 0-connected and $\pi_k(X) = 1$ for $1 \leq k \leq n$. A space X is 0-*connected* if and only if it is path-connected. We say that any space is (-2)-*connected* and a space X is (-1)-*connected* if and only if $X \neq \varnothing$.

After Brown studied the Euler characteristic of the coset poset he wanted to go further and study its homotopy type. His main results are the following.

Proposition 2.3. (See [Bro00, Proposition 8])
Let G be a finite group and N a normal subgroup of G. If N is contained in the Frattini subgroup, then $\mathscr{C}(G)$ is homotopy equivalent to $\mathscr{C}(G/N)$.

Proposition 2.4. (See [Bro00, Proposition 10])
Let G be a finite group and N a normal subgroup of G. Then

$$j\colon \mathscr{C}(G) \to \mathscr{C}(G/N) * \mathscr{C}(G,N)$$

is a homotopy equivalence, where $\mathscr{C}(G,N)$ denotes the set of all cosets of $\mathscr{C}(G)$ which surject onto G/N under the quotient map.

Ramras states in [Ram05] that Proposition 2.4 also holds if G is an infinite group.

Proposition 2.5. (See [Bro00, Proposition 11])
Let G be a finite solvable group and let

$$1 = N_0 < N_1 < ... < N_k = G$$

be a chief series. Then $\mathscr{C}(G)$ has the homotopy type of a bouquet of $(d-1)$-spheres, where d is the number of indices $i = 1, ..., k$ such that N_i/N_{i-1} has a complement in G/N_{i-1}. The number of spheres is $(-1)^{d-1}\tilde{\chi}(\mathscr{C}(G))$.

We will simplify the way of computing the reduced Euler characteristic $\tilde{\chi}(\mathscr{C}(G))$ of Proposition 2.5 in Lemma 2.16.

Brown's paper [Bro00] also raised some questions. One of them is the following. Is there a finite group G such that the coset poset $\mathscr{C}(G)$ is contractible? J. Shareshian and R. Woodroofe answered this in the negative in [SW16].

Theorem 2.6. (See [SW16, Theorem 1.1])
If G is a finite group, then $\Delta\mathscr{C}(G)$ is not contractible.

They used the classification of finite simple groups, the homotopy equivalence of Proposition 2.4 and Smith Theory to prove that $\Delta\mathscr{C}(G)$ is not $\mathbb{F}_2$-acyclic if G is a finite group.

Another question asked by Brown in [Bro00] was considered by D.A. Ramras and is the following. For which finite groups G is $\mathscr{C}(G)$ simply connected? To answer this, Ramras studied the simple connectivity of the coset poset for arbitrary groups in [Ram05]. This led to the following results.

Theorem 2.7. (See [Ram05, Theorem 2.3])
If G is not k-generated, then $\mathscr{C}(G)$ is $(k-1)$-connected.

Corollary 2.8. (See [Ram05, Remark 2.4])
If G is an infinitely generated group, then $\mathscr{C}(G)$ is contractible.

He proved this using atomized posets, theories of posets from [Bjö95], and a simplicial complex with less vertices but higher dimension than the coset poset $\mathscr{C}(G)$.

Little is known about the coset poset of the third class of groups, the class of finitely generated but infinite groups. This motivated us to study the homotopy type of the coset poset of those groups, focusing on the contractibility. We will use different methods than the ones from Shareshian, Woodroofe and Ramras. But as Ramras, instead of studying the homotopy type of the order complex $\Delta\mathscr{C}(G)$ we study another simplicial complex, the nerve complex $\mathcal{NC}(G, \mathscr{H}_\mathscr{C})$. This is a simplicial complex with the same vertex set but of higher dimension.

Definition 2.9. (Nerve Complex, see [AH93, Sections 1.1 and 2.1])
The *nerve complex* $\mathcal{NC}(G, \mathscr{H})$ of a group G with respect to a family $\mathscr{H}$ of subgroups of G is a simplicial complex with an n-simplex for every set $\{H_0g_0, ..., H_ng_n\}$ with $H_i \in \mathscr{H}$ and $g_i \in G$, such that $H_0g_0 \cap ... \cap H_ng_n \neq \varnothing$.

In other literature, as in [BFM+16], the nerve complex is called the coset complex. For more information about the original use of the nerve complex see Section 8.4.

If $C_0 \subset ... \subset C_n$ is a chain with $C_i \in \mathscr{C}(G)$, then $C_0 \cap ... \cap C_n = C_0$ and thus $\{C_0, ..., C_n\}$ is a simplex in $\mathcal{NC}(G, \mathscr{H}_\mathscr{C})$. Therefore $\Delta\mathscr{C}(G)$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_\mathscr{C})$ with the same vertex set.

We call a family $\mathcal{U}$ of subsets of a set $\mathcal{M}$ *closed under intersection* if $\mathcal{U}$ has the following property: if $U, V \in \mathcal{U}$ and $U \cap V \neq \varnothing$, then $U \cap V \in \mathcal{U}$. The next theorem follows directly from [AH93, Theorem 1.4] by Abels and Holz.

Theorem 2.10.
Let $P_\mathscr{H}$ be the vertex set of $\mathcal{NC}(G, \mathscr{H})$. Suppose that $P_\mathscr{H}$ is closed under intersection. Then the nerve complex $\mathcal{NC}(G, \mathscr{H})$ and the order complex $\Delta P_\mathscr{H}$ are homotopy equivalent.

If $\mathscr{H} = \mathscr{H}_\mathscr{C}(G)$, then $P_\mathscr{H} = \mathscr{C}(G)$, which is closed under intersection. Therefore we obtain the following corollary.

Corollary 2.11.
Let G be a group. Then the order complex $\Delta\mathscr{C}(G)$ and the nerve complex $\mathcal{NC}(G, \mathscr{H}_\mathscr{C})$ are homotopy equivalent.

Recall that in the notation of the nerve complex $\mathcal{NC}(G, \mathscr{H})$ the set $\mathscr{H}$ is always a set of subgroups of G and the vertex set of $\mathcal{NC}(G, \mathscr{H})$ is the set of cosets of the subgroups of $\mathscr{H}$ in G. Therefore we write $\mathcal{NC}(G, \mathscr{H}_\mathscr{C})$ instead of $\mathcal{NC}(G, \mathscr{H}_\mathscr{C}(G))$.

Example 2.12. We study the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle\, a, b \,|\, a^2, b^2, aba^{-1}b^{-1} \,\rangle$ of the cyclic group $\mathbb{Z}_2$ with itself. Figure 1 shows the order complex of the coset poset $\mathscr{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in the first row on the left. The order complex $\Delta\mathscr{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a 1-dimensional simplicial complex and the barycentric subdivision of the 1-skeleton of a tetrahedron. On the right side of Figure 1 one sees the nerve complex $\mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_\mathscr{C})$. The nerve complex is built of four 3-simplices and their faces. The 3-simplices share vertices but no other faces and thus create an octahedron shaped hole in the center. As we see $\Delta\mathscr{C}(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_\mathscr{C})$ and both are 0-connected but not 1-connected.

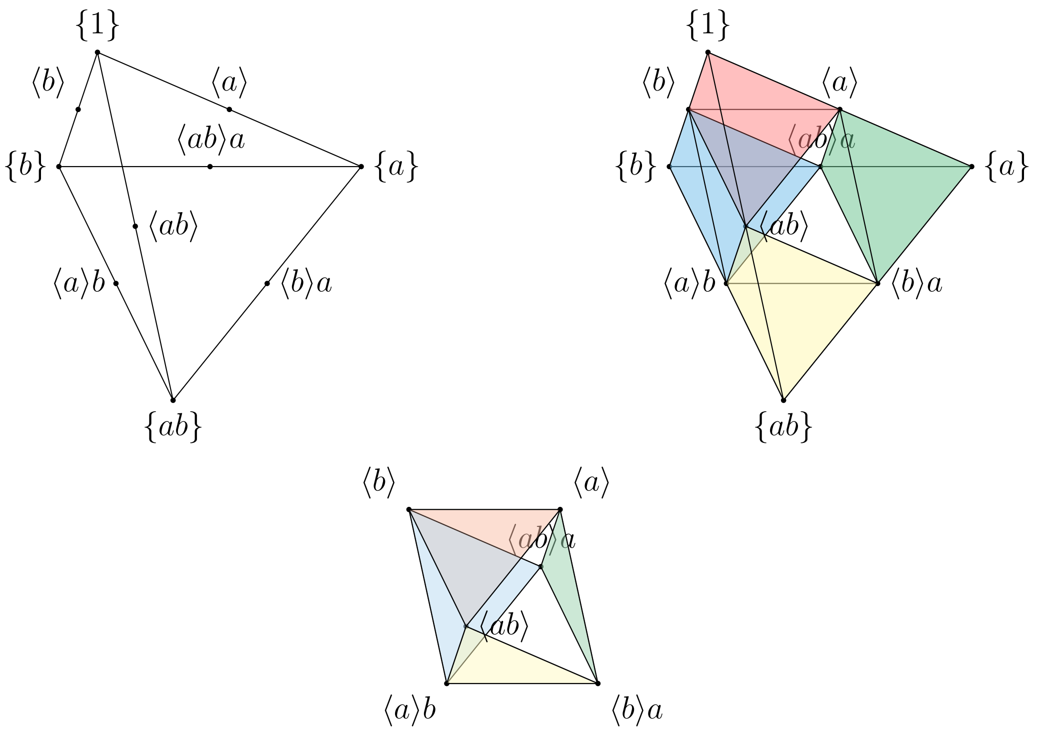

Figure 1: The order complex $\Delta\mathscr{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ at the top left, the nerve complex $\mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_{\mathscr{C}})$ at the top right, and the nerve complex $\mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_{\max})$ at the bottom.

It is easier to check if a set of cosets has a (non-)empty intersection than to check if they create a chain. Thus, if possible, we study nerve complexes instead of order complexes.

Using simplicial collapses the author proved the following.

Proposition 2.13. (See [Wel, Satz 4.14])
Let G be a group, $\mathscr{H}$ and $\mathscr{U}$ families of subgroups of G, and $\mathscr{H}' = \mathscr{H} \cup \mathscr{U}$. Suppose that $\mathcal{NC}(G, \mathscr{H}')$ is a finite nerve complex and for all $H' \in \mathscr{U}$ there exists at least one $H \in \mathscr{H}$ such that $H' \leq H$. Then $\mathcal{NC}(G, \mathscr{H})$ and $\mathcal{NC}(G, \mathscr{H}')$ are homotopy equivalent.

For a group G we define

$$\mathscr{H}_{\max}(G) := \{M < G \mid M \text{ maximal subgroup of } G\}. \tag{2.0.3}$$

Thus we obtain the following corollary.

Corollary 2.14.
Let G be a finite group. Then the nerve complexes $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and $\mathcal{NC}(G, \mathscr{H}_{\max})$ are homotopy equivalent. Therefore the order complex $\Delta\mathscr{C}(G)$ is homotopy equivalent to $\mathcal{NC}(G, \mathscr{H}_{\max})$.

Example 2.15. We consider the nerve complex $\mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_{\max})$, which is depicted in the second row of Figure 1, with $\mathscr{H}_{\max} = \{\langle a\rangle, \langle b\rangle, \langle ab\rangle\}$. This is a 2-dimensional simplicial complex built of four 2-simplices, which only share vertices but no edges. The nerve complex $\mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_{\max})$ is 0-connected and homotopy equivalent to the other simplicial complexes pictured in Figure 1.

We have to be careful with Corollary 2.14. It does not state that $\Delta\mathscr{C}(G)$ is homotopy equivalent to the order complex $\Delta\mathscr{C}_{\max}(G)$ with the same vertex set as $\mathcal{NC}(G, \mathscr{H}_{\max})$. In fact, $\Delta\mathscr{C}_{\max}(G) = (\mathcal{NC}(G, \mathscr{H}_{\max}))^0$ and therefore at most (-1)-connected. For example, $\Delta\mathscr{C}_{\max}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a simplicial complex consisting of 6 unconnected vertices.

Since the simplicial complexes $\Delta\mathscr{C}(G)$, $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$, and $\mathcal{NC}(G, \mathscr{H}_{\max})$ are homotopy equivalent, the Euler characteristics $\chi(\Delta\mathscr{C}(G))$, $\chi(\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}}))$, and $\chi(\mathcal{NC}(G, \mathscr{H}_{\max}))$ are equal. Thus we can choose the simplicial complex whose number of simplices can be computed in the easiest way. If the maximal subgroups are known, this is often $\mathcal{NC}(G, \mathscr{H}_{\max})$, since the vertex set of that complex is smaller than the vertex set of the others. Thus we can rewrite Proposition 2.5 in the following way.

Lemma 2.16.
Let G be a finite solvable group. Then the order complex $\Delta\mathscr{C}(G)$ and the nerve complexes $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and $\mathcal{NC}(G, \mathscr{H}_{\max})$ have the homotopy type of a wedge of $(d-1)$-spheres. The number of spheres is

$$(-1)^{d-1}\tilde{\chi}(\mathcal{NC}(G, \mathscr{H}_{\max})).$$

Example 2.17. We compute the Euler characteristic of the simplicial complexes pictured in Figure 1.

$$\chi(\Delta\mathscr{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)) = k_0 - k_1 = 10 - 12 = -2$$

$$\chi(\mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_{\mathscr{C}})) = k_0 - k_1 + k_2 - k_3 = 10 - 24 + 16 - 4 = -2$$

$$\chi(\mathcal{NC}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathscr{H}_{\max})) = k_0 - k_1 + k_2 = 6 - 12 + 4 = -2$$

Moreover, with the chief series $1 < \mathbb{Z}_2 < \mathbb{Z}_2 \times \mathbb{Z}_2$ we obtain $d = 2$ and thus $(-1)^{d-1}\tilde{\chi}(\mathcal{NC}(G, \mathscr{H}_{\max})) = -(-2-1) = 3$. Therefore $\mathscr{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has the homotopy type of a wedge of 3 spheres $\mathbb{S}_1$ of dimension 1.

Using the result that if $\mathscr{H}$ is finite, the maximal simplices in $\mathcal{NC}(G, \mathscr{H})$ are of dimension $|\mathscr{H}| - 1$, the author proved the following proposition.

Proposition 2.18. (See [Wel, Satz 4.7])
Let G be a finitely generated group. If $\mathscr{H}$ is a finite set and $\mathcal{NC}(G, \mathscr{H})$ a finite simplicial complex, then $\mathcal{NC}(G, \mathscr{H})$ is either contractible or at most $(|\mathscr{H}| - 2)$-connected.

With Theorem 2.6 we obtain the following.

Corollary 2.19.
Let G be a finite group. Then the nerve complexes $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and $\mathcal{NC}(G, \mathscr{H}_{\max})$ and therefore the order complex $\Delta\mathscr{C}(G)$ are at most $(|\mathscr{H}_{\max}(G)| - 2)$-connected.

Thus we developed an upper bound of the connectivity of the coset poset of a finite group in addition to the lower bound given by Ramras. Therefore we are interested in the following result.

Proposition 2.20.
For every $n \in \mathbb{N}$, there exists a finite group G_n such that the coset poset $\mathscr{C}((\mathbb{Z}_p)^{n+2})$ is n-connected but not $(n+1)$-connected. For example we may put $G_n = (\mathbb{Z}_p)^{n+2}$, with p prime.

Proof.

$$\{1\} < \mathbb{Z}_p < \mathbb{Z}_p \times \mathbb{Z}_p < (\mathbb{Z}_p)^3 < ... < (\mathbb{Z}_p)^{n+1} < (\mathbb{Z}_p)^{n+2}$$

is a chief series of the solvable group $(\mathbb{Z}_p)^{n+2}$. Since each factor $(\mathbb{Z}_p)^i/(\mathbb{Z}_p)^{i-1}$ with $i = 1, ..., n+2$ has a complement in $(\mathbb{Z}_p)^{n+2}/(\mathbb{Z}_p)^{i-1}$, we have $d = n+2$. By Proposition 2.5, $\mathscr{C}((\mathbb{Z}_p)^{n+2})$ has the homotopy type of a wedge of $(n+1)$-spheres. Consequently, $\mathscr{C}((\mathbb{Z}_p)^{n+2})$ is n-connected but not $(n+1)$-connected. □

After studying the coset poset of finite groups, we now turn to finitely generated infinite groups.

Part II

Subgroup graphs

After considering the coset poset of finite groups we turn to finitely generated but infinite groups. To examine the coset poset of such groups we study a subset, namely the finite index coset poset. We will prove that the finite index coset poset as well as the coset poset is contractible for many finitely generated infinite groups by proving the existence of infinitely many special maximal subgroups. Therefore we need a tool to prove the existence of such groups. This tool is our theory of subgroup graphs, which we introduce in this part. It is interesting by itself and already led to a publication [Wel17].

This part is organized as follows. In Section 3 we recall the theory of subgroup graphs of subgroups of free groups by Ilya Kapovich and Alexei Myasnikov [KM02]. This theory led to our theory of subgroup graphs of finite index subgroups of finitely generated groups, which we develop in Section 4. In Section 5, we state applications of our theory of subgroup graphs of finite index subgroups. (Sections 3–5 are equal to [Wel17, Sections 2–4] except for Section 5.6.)

3 Subgroup graphs of subgroups of free groups

Following the article [KM02], this section is meant to recall the theory of subgroup graphs in free groups from Kapovich and Myasnikov. Their approach is more combinatorial and computational than the topological one by J. Stallings [Sta83]. Both approaches study subgroups of free groups. We start by defining the terms X-graph, language of an X-graph, folded graph, and core graph, see Definitions 3.1, 3.5, 3.7 and 3.10, respectively. Then we recall the subgroup graph theory of [KM02]. Let $F(X)$ be a finitely generated free group. To a based folded connected X-graph (Γ, v), which is a core graph with respect to v, they associate a subgroup $H \leq F(X)$, by taking the language $L(\Gamma, v) = H$, see Lemma 3.9. Conversely, for every subgroup $H \leq F(X)$ there exists a based X-graph (Γ', v'), unique up to isomorphism, which is connected, folded, and a core graph with respect to v' such that its language $L(\Gamma', v')$ is H. The correspondence is unique up to isomorphism of based X-graphs. Therefore we call (Γ', v') the subgroup graph $\Gamma_X(H)$ of $H \leq F(X)$ with base-vertex $v' = 1_H$. This is stated in Theorem 3.11. The graph $\Gamma_X(H)$ is the core of the Schreier coset graph of H in $F(X)$ with respect to X at the base-vertex H, see Definition 3.13 and our Remark 3.12. If H is a subgroup of finite index in $F(X)$, then $\Gamma_X(H)$ is the Schreier coset graph of H with respect to $F(X)$. One of the most important parts for our theory of subgroup graphs of finite index subgroups, which we state in the next section, is the result of Proposition 3.18. It states that the index of the subgroup H is finite if and only if the subgroup graph $\Gamma(H)$ is a finite X-regular graph. Moreover, the number of vertices of $\Gamma(H)$ is the index of H in G.

Definition 3.1. (X-Graph, see [KM02, Definition 2.1])
Let X be a finite set which is called an *alphabet*. Let Γ be a (finite or infinite) directed multi-edge graph with vertex set $V(\Gamma)$ and set of directed edges $E(\Gamma)$. We denote by $o(e)$ the *origin* and by $t(e)$ the *terminus* of the edge e. We say e is an edge from $o(e)$ to $t(e)$. If $o(e) = t(e)$ then e is called a *loop*.

A graph Γ is called an *X-labeled directed graph* (or *X-digraph*, or *X-graph*) if every directed edge $e \in E(\Gamma)$ is labeled by a letter from X, which is denoted by $\mu(e)$.

A map $\pi \colon \Gamma \to \Gamma'$, between two X-graphs is called a *morphism* of X-graphs if π takes vertices to vertices, directed edges to directed edges, preserves labels of directed edges and has the property that $o(\pi(e)) = \pi(o(e))$, $t(\pi(e)) = \pi(t(e))$ for every edge e of Γ. Furthermore, a morphism of based X-graphs (Γ, v), (Γ', v') maps v to v'.

In this thesis X^{-1} will always denote the set of all formal inverses of the elements in X and X will always be finite.

Definition 3.2. (See [KM02, Convention 2.2])
Let Γ be an X-graph. We define the *$(X \cup X^{-1})$-graph* $\widehat{\Gamma}$ as follows: $V(\widehat{\Gamma}) := V(\Gamma)$ and for every edge $e \in E(\Gamma)$ we introduce the formal inverse e^{-1} of e, whose label is $\mu(e)^{-1}$. The endpoints of e^{-1} are $o(e^{-1}) := t(e)$ and $t(e^{-1}) := o(e)$. For a new edge e^{-1}, we set $(e^{-1})^{-1} = e$. We have that $E(\widehat{\Gamma}) := E(\Gamma) \cup E(\Gamma)^{-1}$. For an example see Figure 2.

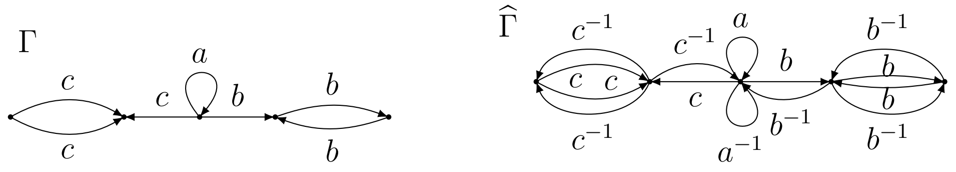

Figure 2: An X-graph Γ and the $X \cup X^{-1}$-graph $\widehat{\Gamma}$ for $X = \{a, b, c\}$.

The edges e and e^{-1} are usually drawn as a single directed edge e with understanding that one can pass it in a negative direction to obtain e^{-1}. Hence one can abuse the notation and disregard the difference between Γ and $\hat{\Gamma}$

Definition 3.3. (Path, see [KM02, Convention 2.2])
Let Γ be an X-graph. A *path* p in Γ is, by definition, a sequence of edges $e_1, ..., e_k$, where $e_i \in E(\widehat{\Gamma})$ and $o(e_i) = t(e_{i-1})$. The origin of p is $o(p) := o(e_1)$ and its terminus is $t(p) := t(e_k)$. The label of p is, by definition, $\mu(p) := \mu(e_1) \cdots \mu(e_k)$, the word in the free monoid generated by $X \cup X^{-1}$. We call p a path from $o(e_1)$ to $t(e_k)$. If v is a vertex of Γ, we consider the sequence $p = v$ to be a path with $o(p) = t(p) = v$ and $\mu(p) = 1$ (the empty word).

Definition 3.4. (Reduced Word and Reduced Path, see [KM02, Convention 2.6])
A *freely reduced word* in the alphabet $X \cup X^{-1}$ is a word without any subwords xx^{-1} or $x^{-1}x$ for $x \in X$. A path p in an X-graph Γ is said to be *reduced* if p does not contain subpaths of the form e, e^{-1} for $e \in E(\widehat{\Gamma})$.

Let w be a word in the alphabet $X \cup X^{-1}$ (or an X-word, or a word in $X^{\pm 1}$). We denote by $\overline{w}$ the freely reduced X-word obtained by removing xx^{-1}, $x^{-1}x$ successively. *The free group* on X, denoted $F(X)$, is the collection of all freely reduced words in $X^{\pm 1}$. The multiplication in the free group $F(X)$ is defined as

$$f \cdot g := \overline{fg}$$

for all $f, g \in F(X)$.

Definition 3.5. (Language, see [KM02, Definition 2.7])
Let Γ be an X-graph and let v be a vertex of Γ. We define the *language of* Γ *with respect to* v to be

$$L(\Gamma, v) = \{\mu(p) \mid p \text{ is a reduced path in } \Gamma \text{ with } o(p) = t(p) = v\}.$$

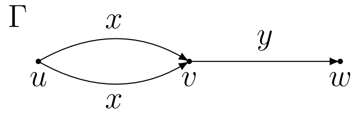

Figure 3: An $\{x, y\}$-graph Γ.

For example, in Figure 3 the languages of Γ with respect to the vertices u, v or w are: $L(\Gamma, u) = \{(xx^{-1})^n \mid n \in \mathbb{N}\}$, $L(\Gamma, v) = \{(x^{-1}x)^n \mid n \in \mathbb{N}\}$ and $L(\Gamma, w) = \{1, y^{-1}(x^{-1}x)^n y \mid n \in \mathbb{N}_{>0}\}$.

Note that $\mu(p)$ may have subwords of the form xx^{-1} or $x^{-1}x$ for some $x \in X$ even if p is a reduced path. Hence the words in the language $L(\Gamma, v)$ of an X-graph are not necessarily freely reduced.

Proposition 3.6. (See [KM02, Proposition 3.1])
Let Γ be an X-graph and let v be a base-vertex of Γ. Then the set

$$\overline{L(\Gamma, v)} = \{\overline{w} \mid w \in L(\Gamma, v)\}$$

is a subgroup of the free group $F(X)$.

As we will see, the language of a folded X-graph consists only of freely reduced words.

Definition 3.7. (Folded X-Graph, see [KM02, Definition 2.3])
Let Γ be an X-graph. We say that the X-graph Γ is *folded* if for each vertex v of Γ and each letter $x \in X$ there is at most one edge in Γ with origin v and label x and at most one edge with terminus v and label x.

For example, in Figure 4, the graphs Γ, Γ' and Γ'' are not folded but the graph Γ''' is folded.

Suppose Γ is an X-graph and e, e' are two edges of Γ with $o(e) = o(e')$ (or $t(e) = t(e')$) and the same label $x \in X$. Then, informally speaking, *folding* Γ at e, e' means identifying e and e' in a single new edge with label x. For a more precise definition see [KM02, Definition 2.4] and [Sta83, Section 3.2].

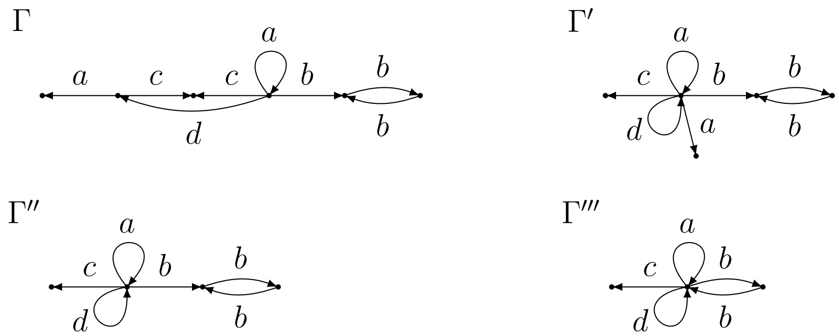

Figure 4: Folding of an $\{a, b, c, d\}$-graph Γ. Every step $\Gamma \dashrightarrow \Gamma'$, $\Gamma' \dashrightarrow \Gamma''$, and $\Gamma'' \dashrightarrow \Gamma'''$ is a folding.

With the definition of a folded X-graph at hand we may associate a subgroup to the language of an X-graph.

Lemma 3.8. (See [KM02, Lemma 2.9])
Let Γ be a folded X-graph and v be a vertex of Γ. Then all the words in the language $L(\Gamma, v)$ are freely reduced.

Lemma 3.9. (See [KM02, Lemma 3.2])
Suppose Γ is a folded X-graph. Then $L(\Gamma, v) = \overline{L(\Gamma, v)}$ is a subgroup of the free group $F(X)$.

To each folded based X-graph (Γ, v) we have thus associated a subgroup of the free group by considering the language $L(\Gamma, v)$ of the X-graph. But two different folded based X-graphs can have the same language. For example, each of these four based graphs (Γ, u), (Γ, v), (Γ', v') and (Γ', u') in Figure 5 has language $\{x^{2z} \mid z \in \mathbb{Z}\}$.

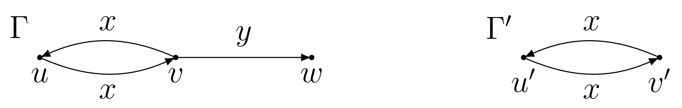

Figure 5: An X-graph Γ and an X-graph $\Gamma' = Core(\Gamma, v)$ for $X = \{x, y\}$.

Definition 3.10. (Core Graph, see [KM02, Definition 3.5])
Let Γ be an X-graph and let v be a vertex. Then the *core of* Γ *at* v is defined as

$$Core(\Gamma, v) = \bigcup \{p \mid p \text{ is a reduced path in } \Gamma \text{ with } o(p) = t(p) = v\}.$$

If $Core(\Gamma, v) = \Gamma$ we say that Γ is a *core graph with respect to* v.

For example, the graph Γ in Figure 5 is a core graph only with respect to w. For the vertex u and v we have that $Core(\Gamma, u) = Core(\Gamma, v) = \Gamma'$.

The next theorem is a combination of [KM02, Theorem 5.1],[KM02, Theorem 5.2] and [KM02, Definition 5.3].

Theorem 3.11. (Subgroup Graph, see [KM02])
Let $H \leq F(X)$ be a subgroup. There exists a based X-graph (Γ, v) (unique up to canonical isomorphism of based X-graphs) such that

(i) the graph Γ is folded and connected;

(ii) the graph Γ is a core graph with respect to v;

(iii) $L(\Gamma, v) = H$.

In this situation we call Γ the subgroup graph of H with respect to X and denote it by $\Gamma_X(H)$ or briefly by $\Gamma(H)$. The base-vertex v is denoted 1_H.

Remark 3.12. The subgroup graph $\Gamma_X(H)$ is the core of the Schreier coset graph of H with respect to X and $F(X)$, see the proof of [KM02, Theorem 5.1].

By Theorem 3.11, the graph $\Gamma(H)$ is unique (up to isomorphism). Consequently, if (Γ, v) is a folded connected core X-graph with language $L(\Gamma, v) = H$, then $(\Gamma, v) \cong (\Gamma(H), 1_H)$.

Definition 3.13. (Schreier Coset Graph)
Let G be a group with a finite generating set X. Let H be a subgroup of G. Let Γ be the following X-graph. The vertex set of Γ is the set of right cosets of the subgroup H in G. For two cosets Hg and Hg' and each letter $x \in X$ we introduce a directed edge with origin Hg, terminus Hg' and label x whenever $Hgx = Hg'$. This graph is called the *Schreier coset graph* of H with respect to X and G.

There are three important classes of subgroups of the free group $F(X)$: finite index subgroups, finitely generated subgroups and infinitely generated ones. In their article [KM02], Kapovich and Myasnikov show that each of these three classes have subgroup graphs with specific properties.

Lemma 3.14. (See [KM02, Lemma 5.4])
For a subgroup $H \leq F(X)$ the subgroup graph $\Gamma(H)$ is finite if and only if H is finitely generated.

The next notion distinguishes finitely generated subgroups of $F(X)$ from those of finite index (which are also finitely generated).

Definition 3.15. (Regular X-Graph, see [KM02, Definition 8.1])
An X-graph Γ is said to be *X-regular* if for every vertex v of Γ and every $x \in X \cup X^{-1}$ there is exactly one edge in $\widehat{\Gamma}$ with origin v and label x.

Figure 6 contains the following examples. The graph Γ is $\{a, b\}$-regular, the graph Γ' and Γ'' are not $\{a, b\}$-regular and the graph Γ'' is $\{a\}$-regular.

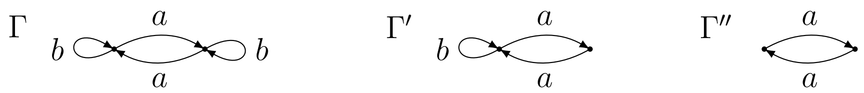

Figure 6: $\{a, b\}$-regular and not $\{a, b\}$-regular graphs.

We can reformulate the notion of being X-regular. An X-graph Γ is X-regular if for every vertex v of Γ and every $x \in X$ there is exactly one edge with label x and origin v and exactly one edge with label x and terminus v.

By the above reformulation of X-regularity and the definition of folded X-graphs we obtain:

Lemma 3.16.
An X-regular graph is folded.

Lemma 3.17.
A connected finite X-regular graph Γ is a core graph with respect to every vertex of $V(\Gamma)$.

Proposition 3.18. (See [KM02, Proposition 8.3])
Let H be a subgroup of the free group $F(X)$. Then the index $[F(X) : H]$ is finite if and only if the subgroup graph $\Gamma(H)$ is a finite X-regular graph. In this case $[F(X) : H] = |V(\Gamma(H))|$.

The subgroup graph $\Gamma(H)$ of H is the core of the Schreier coset graph of H with respect to X and $F(X)$. If H is a finite index subgroup of $F(X)$, then the Schreier coset graph of H with respect to X and $F(X)$ is finite and X-regular and thus a core graph. Hence the subgroup graph $\Gamma(H)$ and the Schreier coset graph of H are equal if H has finite index in $F(X)$.

4 Subgroup graphs of finite index subgroups of finitely generated groups

In this section we generalize the theory of subgroup graphs of subgroups of free groups, as described in Section 3, to finite index subgroups of finitely generated groups. Suppose that G is a finitely generated group with a presentation $\langle X \mid R \rangle$, where X is finite and R is not necessarily finite. To modify Kapovich and Myasnikov's theory of subgroup graphs of free groups to a theory of subgroup graphs of finitely generated groups we need a condition which ensures the uniqueness of the subgroup graphs. Therefore we introduce the term fulfilling the defining relators in Definition 4.1. To a finite connected based X-regular graph (Γ, v) which fulfills the defining relators R, we associate a finite index subgroup $H \leq G$, by taking $H := \phi(L(\Gamma, v))$. Conversely, for every finite index subgroup $H \leq G$ there exists a finite connected X-regular based graph (Γ', v'), unique up to isomorphism, which fulfills the defining relators R and has $\phi(L(\Gamma', v')) = H$. We prove this in Theorem 4.5. The correspondence is unique up to isomorphism of based X-graphs. Therefore we call (Γ', v') the subgroup graph $\Gamma_{X,R}(H)$ of $H \leq G$. It is the Schreier coset graph of H with respect to X and G. Notice that the subgroup graph $\Gamma_{X,R}(H)$ depends on the presentation of the group G. Therefore the subgroup graphs $\Gamma_{X,R}(H)$ and $\Gamma_{X',R'}(H)$ of H may not be isomorphic for different generators X, X' and relators R, R', as we see in Remark 5.7. However, the number of vertices $|V(\Gamma_{X,R}(H))| = [G : H]$ is independent of X and R.

For the rest of this thesis, let us fix some notation: Let $G = \langle X \mid R \rangle$ be a finitely generated group, where X is finite and R is a subset of words of the free group $F(X)$, not necessarily finite. We call X the *set of generators* and R the *set of relators*. Denote by $N := \langle\!\langle R \rangle\!\rangle_{F(X)} = \left\langle \overline{wrw^{-1}} \mid w \in F(X), r \in R \right\rangle$ the normal closure of R in $F(X)$. Let $\phi\colon F(X) \to G$ be the canonical epimorphism such that $G \cong F(X)/N$.

Our aim is to develop a theory of subgroup graphs of finite index subgroups of finitely generated groups. A natural way to define a subgroup graph of $H \leq G$ is to use the subgroup graph $\Gamma_X(H')$ of $H' = \phi^{-1}(H) \leq F(X)$. Then $\phi(L(\Gamma_X(H'), 1_{H'})) = H$ and $N \leq H'$. Moreover, if $[F(X) : H'] = n$, then $[G : H] = n$. Since H' is a finite index subgroup in $F(X)$, the subgroup graph $\Gamma_X(H')$ is finite, connected and X-regular. If we choose a subgroup graph $\Gamma_X(H'')$ of $H'' < F(X)$ such that $\phi(H'') = H$, then $H'' \subseteq \phi^{-1}(H)$. Therefore there can be different subgroups of $F(X)$ with H as an image under ϕ. But if $N \leq L(\Gamma_X(H''), 1_{H''}) \leq F(X)$, then $L(\Gamma_X(H''), 1_{H''}) = \phi^{-1}(H)$. Consequently, to determine the subgroup graph $\Gamma_{X,R}(H)$, we need a condition which ensures that $N \leq L(\Gamma_{X,R}(H), v)$.

Definition 4.1. (Fulfilling X-Graph)
Let $G = \langle X \mid R \rangle$ be a presentation of a group, where X is finite and R is not necessarily finite. We say that an X-graph Γ *fulfills the defining relators* $r \in R$, or briefly Γ fulfills R, if for all vertices $v \in V(\Gamma)$ the following holds: if p_r is a reduced path with origin v and label $r \in R$, then the terminus of p_r is v.

Example 4.2. We consider the graphs of Figure 7. The $\{x, y\}$-regular graph Γ fulfills the relators x^2, y^2 and $(xy)^3$. Indeed, the reduced paths with label x^2 are: $v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_1$; $v_2 \xrightarrow{x} v_1 \xrightarrow{x} v_2$; $v_3 \xrightarrow{x} v_3 \xrightarrow{x} v_3$. For the relator $(xy)^3$, an example of a reduced path with that label is $v_1 \xrightarrow{x} v_2 \xrightarrow{y} v_3 \xrightarrow{x} v_3 \xrightarrow{y} v_2 \xrightarrow{x} v_1 \xrightarrow{y} v_1$. Notice that the graph Γ does not fulfill the relator $(xy)^2$. In fact, the reduced path with label $(xy)^2$ and origin v_1 has terminus v_2.

Instead, the $\{x, y\}$-regular graph Γ' fulfills the relator $(xy)^2$. The reduced paths with label $(xy)^2$ are: $u_1 \xrightarrow{x} u_2 \xrightarrow{y} u_1 \xrightarrow{x} u_2 \xrightarrow{y} u_1$; $u_2 \xrightarrow{x} u_3 \xrightarrow{y} u_3 \xrightarrow{x} u_4 \xrightarrow{y} u_2$; $u_3 \xrightarrow{x} u_4 \xrightarrow{y} u_2 \xrightarrow{x} u_3 \xrightarrow{y} u_3$; $u_4 \xrightarrow{x} u_1 \xrightarrow{y} u_4 \xrightarrow{x} u_1 \xrightarrow{y} u_4$. The graph Γ' also fulfills the relators x^4 and y^3.

Note that if a graph fulfills a relator w^n, then it also fulfills the relator w^{nz} for $z \in \mathbb{Z}$.

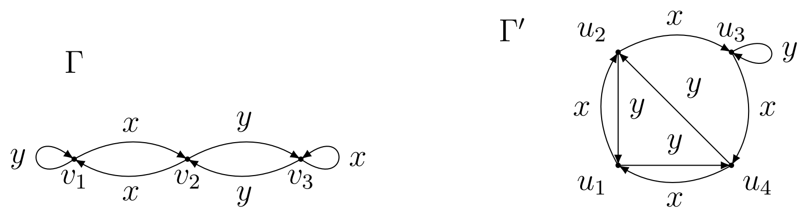

Figure 7: Two $\{x, y\}$-graphs which fulfill different relators.

Remark 4.3. Let p_v be a reduced path from 1_H to v in $\Gamma_X(H)$ and $\mu(p_v) = g_v$. The subgroup graph $\Gamma_X(H)$ of a finite index subgroup H of the free group $F(X)$ is the Schreier coset graph of H. Therefore $H\backslash F(X) = \{Hg_v \mid v \in V(\Gamma_X(H))\}$. Moreover,

$$Hg_v = \{\mu(p) \mid p \text{ is a reduced path with } o(p) = 1_H \text{ and } t(p) = v\}.$$

Using the above remark we can prove the following proposition.

Proposition 4.4.
Let Γ be a finite X-regular connected graph with base-vertex v_0. Then Γ fulfills the defining relators R if and only if $\langle\langle R \rangle\rangle_{F(X)} \leq L(\Gamma, v_0)$.

Proof. Since Γ is a finite connected X-regular graph, Γ is also folded and a core graph, by Lemmas 3.16 and 3.17. By Lemma 3.9, the language $H = L(\Gamma, v)$ is a subgroup of $F(X)$. Following Theorem 3.11, Γ is the subgroup graph $\Gamma_X(H)$ of H, where $v_0 = 1_H$. By Proposition 3.18, the subgroup H has finite index in $F(X)$ and $\Gamma_X(H)$ is a Schreier coset graph.

Assume that Γ fulfills the relators R. Let w be in $F(X)$. Since Γ is X-regular, there exists a unique reduced path p_w with $\mu(p_w) = w$ and $o(p_w) = v_0$. Let v be the terminus of the path p_w. Since Γ fulfills the defining relators R, the reduced path p_r with $\mu(p_r) = r \in R$ and $o(p_r) = v$ has terminus v. The path p_w^{-1} with $\mu(p_w^{-1}) = w^{-1}$ and $o(p_w^{-1}) = v$ ends in v_0. Consequently, the path $p_w p_r p_w^{-1}$ is a path from v_0 to v_0, which may not be reduced. Let p be the path which we obtain by deleting successively all subpaths ee^{-1} from $p_w p_r p_w^{-1}$, where $e \in E(\widehat{\Gamma})$.

Then the path p is reduced. By definition $\mu(p) \in L(\Gamma, v_0)$. Since the graph Γ is X-regular, $\mu(p)$ is a freely reduced word. Therefore $\mu(p) = \overline{wrw^{-1}} \in L(\Gamma, v_0)$. Hence $\langle\!\langle R \rangle\!\rangle_{F(X)} \leq L(\Gamma, v_0)$.

Assume now that $\langle\!\langle R \rangle\!\rangle_{F(X)} \leq L(\Gamma, v_0) = H \leq F(X)$ and let v be a vertex of Γ and $r \in R$. Since Γ is X-regular, there exists a unique reduced path p_r with $\mu(p_r) = r$ and $o(p_r) = v$. Let v' be the terminus of p_r. Let w be the label of a reduced path from v_0 to v. The graph Γ is a Schreier coset graph. Thus the vertex v is the coset Hw and the vertex v' is the coset Hw' with $w' := \overline{wr}$. By assumption, we have $\overline{wrw^{-1}} \in H$, hence $H\overline{wrw^{-1}} = H$ which is equivalent to $Hw' = H\overline{wr} = H\overline{w} = Hw$, hence $v = v'$. □

Finally, we can state the main theorem of this section. Part (1) is the analogue of Lemma 3.9 and part (2) the analogue of Theorem 3.11.

Theorem 4.5. (Subgroup Graph)
Let G be a group with a presentation $G = \langle X \mid R \rangle$, where X is finite and R is not necessarily finite.

(1) Let Γ be an X-regular connected graph with n vertices. Let v_0 be a base-vertex of Γ. Assume that Γ fulfills the defining relators R. Then $H := \phi(L(\Gamma, v_0))$ is a subgroup of G of index $[G : H] = n$.

(2) Let $H \leq G$ be a subgroup of index $[G : H] = n \in \mathbb{N}$. Then there exists a based X-graph (Γ, v_0) (unique up to a canonical isomorphism of based X-graphs) such that

(i) Γ is X-regular and connected;

(ii) Γ fulfills the defining relators R;

(iii) Γ has n vertices;

(iv) $\phi(L(\Gamma, v_0)) = H$.

We call the graph (Γ, v_0) the subgroup graph of H with respect to X and R. We denote it by $\Gamma_{X,R}(H)$ or briefly by $\Gamma(H)$. The base-vertex v_0 is denoted by 1_H. In fact, $\Gamma_{X,R}(H) = \Gamma_X(H')$, where $H' := L(\Gamma_{X,R}(H), 1_H) \leq F(X)$.

Proof. Let us prove part (1). The graph (Γ, v_0) is the subgroup graph of the subgroup $L(\Gamma, v_0) =: H' \leq F(X)$. Since Γ is X-regular and has n vertices, H' is a subgroup of index n in $F(X)$. The graph Γ fulfills the defining relators R. Thus $N \leq H'$, and consequently $[G : \phi(H')] = [F(X) : H'] = n$.

Let us prove part (2). Let $\phi^{-1}(H) =: H' \leq F(X)$. By Theorem 3.11, there is a subgroup graph $\Gamma_X(H')$ for H' (unique up to isomorphism of based X-graphs) which is folded, connected, a core graph and has $L(\Gamma_X(H'), 1_{H'}) = H'$. Thus $\phi(L(\Gamma_X(H'), 1_{H'})) = \phi(H') = \phi(\phi^{-1}(H)) = H$. But $[G : H] = [F(X) : H']$, since $N = \phi^{-1}(1_G) \leq \phi^{-1}(H) = H'$. By Proposition 3.18, $\Gamma_X(H')$ is an X-regular graph with n vertices. By Proposition 4.4, Γ fulfills the defining relators R. Thus $(\Gamma_{X,R}(H), 1_H) = (\Gamma, v_0) \cong (\Gamma_X(H'), 1_{H'})$. □

By the proof of [KM02, Theorem 5.1], we see that the subgroup graph $\Gamma(H)$ of a finite index subgroup H is the Schreier coset graph of H with respect to X and G. Using the Todd-Coxeter algorithm one can find the Schreier coset graph, but this is hard. If we know a finite set of generators for a subgroup H' of $F(X)$, then we can construct the subgroup graph $\Gamma(H')$ of H' easily. This is done by an algorithm with finitely many steps, see [KM02, Proposition 7.1]. Hence if the preimage H_0 of H is known, that is, if it is given by a finite generating set, then there is an easy way to construct the subgroup graph $\Gamma(H)$. The problem is to find the preimage $H_0 = \phi^{-1}(H)$.

Nevertheless, the subgroup graph is a useful tool for working with subgroups. Therefore the important part of Theorem 4.5 is part (1). It shows that if we construct a finite connected X-regular graph which fulfills the defining relators R, then the image of the language is a finite index subgroup. We use this, in Section 6, to prove the contractibility of order and nerve complexes. To create an X-regular graph is not that hard and testing if it fulfills the defining relators R is possible in finitely many steps if R is finite.

Remark 4.6. As $\Gamma(H)$ is the Schreier coset graph for the finite index subgroup $H \leq G$, we obtain the cosets of $H \backslash G$ from the subgroup graph $\Gamma(H)$. Let p_v be a reduced path from 1_H to $v \in V(\Gamma(H))$ and $\phi(\mu(p_v)) = g_v \in G$. Then

$$Hg_v = \{\phi(\mu(p)) \mid p \text{ is a reduced path with } o(p) = 1_H \text{ and } t(p) = v\}.$$

If $g \in Hg_v$, then every X-word w representing g is the label of a path from 1_H to v. Therefore we sometimes write $w \in G$, meaning $\phi(w) \in G$.

We end this section with examples of subgroup graphs of finite index subgroups. As a finite group has only finite index subgroups, our theory gives us all subgroups of a finite group.

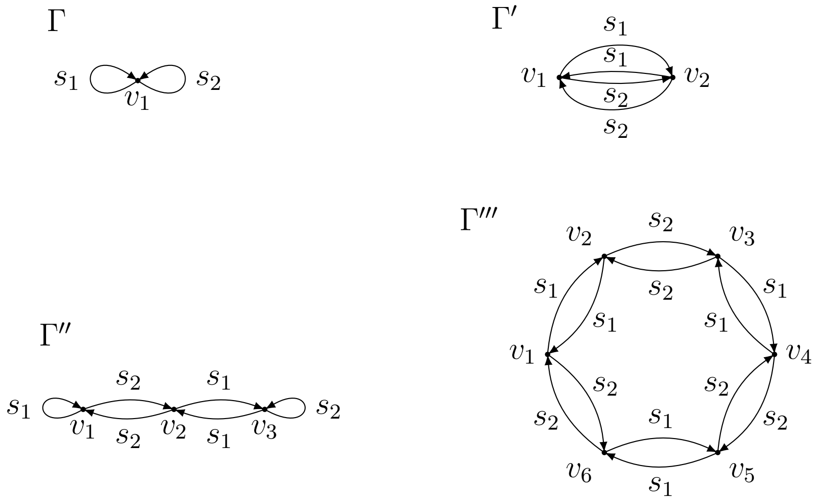

Figure 8: All $\{s_1, s_2\}$-regular graphs, which fulfill the defining relators of the presentation $\langle\, s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \,\rangle$ of the symmetric group S_3.

Example 4.7. Figure 8 shows all finite X-regular graphs, which fulfill the defining relators of the presentation $\langle\, s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \,\rangle$ of the symmetric group S_3.

The graph Γ is the subgroup graph $\Gamma(S_3)$ of the symmetric group S_3. The language of Γ' is the same for both of its vertices and $\phi(L(\Gamma', v_1)) = \langle s_1 s_2 \rangle = A_3$. Therefore $\Gamma(A_3) = \Gamma'$ is the subgroup graph of A_3. The graph Γ'' gives us three different subgroups: $H_1 = \phi(L(\Gamma'', v_1)) = \langle s_1 \rangle$, $H_2 = \phi(L(\Gamma'', v_2)) = \langle s_1 s_2 s_1 \rangle$ and $H_3 = \phi(L(\Gamma'', v_3)) = \langle s_2 \rangle$ in S_3. Hence the subgroup graphs of H_1, H_2 and H_3 are $(\Gamma(H_i), 1_{H_i}) = (\Gamma'', v_i)$. The language of the graph Γ''' is the same for all its vertices. Thus the graph Γ''' is the subgroup graph $\Gamma(\{1_{S_3}\})$ of the trivial group.

For more examples, see Section 5, Section 6.2, Appendix B, where we consider a way to construct subgroup graphs, and Appendix C.

5 Applications of subgroup graphs of finite index subgroups

In this section we extend the results of the article [KM02] to applications for finite index subgroups of finitely generated groups. Furthermore, we add some results. Section 5.1 uses the subgroup graph to detect a Hall subgroup of a finite group. In Section 5.2 we use the subgroup graphs of two finite index subgroups to determine if one is a subgroup of the other. Section 5.3 provides a generating system for a finite index subgroup using its subgroup graph. Section 5.4 shows that $\{\phi(L(\Gamma(H), v)) \mid v \in \Gamma(H)\}$ is the conjugacy class of the finite index subgroup $H \leq G$. In Section 5.5 we prove that if $(\Gamma(H), 1_H) \cong (\Gamma(H), v)$ for all $v \in V(\Gamma(H))$, then the finite index subgroup $H \leq G$ is normal in G. Moreover, we determine the normalizer of a finite index subgroup. In Section 5.6 we prove that the connected component of $1_H \times 1_K$ of the product graph $\Gamma(H) \times \Gamma(K)$ is the subgroup graph of the intersection $H \cap K$ of two finite index subgroups H, K of G, see Proposition 5.21. Using $\Gamma(H) \times \Gamma(K)$ we can examine if the intersection $Hg \cap Kg'$ is empty or not. If not, it gives a representative g'' for $(H \cap K)g'' = Hg \cap Kg'$ for all $g, g' \in G$. We show this in Proposition 5.23. Moreover, we prove that $G = HK$ if and only if $[G : H \cap K] = [G : H][G : K]$ for finite index subgroups H, K. In Section 5.7 we prove that a subgroup H of a finite group is malnormal if and only if $L(\Gamma(H) \times \Gamma(H), v \times u) = N$ for all $u \times v$ not in the connected component of $1_H \times 1_K$.

5.1 Hall and Sylow subgroups

First, we consider Hall subgroups. A *Hall subgroup* of a finite group G is a subgroup H whose order is coprime to its index $[G : H]$. All Sylow subgroups are Hall subgroups.

Proposition 5.1. (Hall and Sylow Subgroups)
Let $G = \langle X \mid R \rangle$ be a finite group of order n with X being finite. There exists a Hall subgroup H of order d (hence d and $\frac{n}{d}$ are coprime) if and only if there exists a connected X-regular graph Γ which fulfills the defining relators R and has $m = \frac{n}{d}$ vertices.

Proof. Let H be a Hall subgroup with $|H| = d$ and $\Gamma_{X,R}(H)$ its subgroup graph. Thus $\Gamma_{X,R}(H)$ is connected, X-regular, fulfills the defining relators R and has m vertices. Hence a graph Γ as above exists.

On the other hand, let Γ be as in Proposition 5.1. By Theorem 4.5 (1), the graph (Γ, v) is a subgroup graph of the subgroup $\phi(L(\Gamma, v))$ of order d which has index m in G. Thus there exists a Hall subgroup of order d. □

5.2 Morphisms and subgroups

The first application we extend is the following.

Lemma 5.2. (See [KM02, Lemma 4.1])
Let $\pi\colon \Gamma \to \Gamma'$ be a morphism of X-graphs, let $v \in V(\Gamma)$ and $v' = \pi(v)$. Suppose

that Γ *and* Γ' *are folded* X*-graphs. Put* $K = L(\Gamma, v)$ *and* $H = L(\Gamma', v')$*. Then* $K \leq H \leq F(X)$*.*

We now state the analogue of Lemma 5.2.

Proposition 5.3. (Morphisms and Subgroups)
Let $G = \langle X \mid R \rangle$ *be a group with* X *finite and* R *not necessarily finite. Let* $\pi\colon \Gamma \to \Gamma'$ *be a morphism of* X*-graphs, let* $v \in V(\Gamma)$ *and* $v' = \pi(v)$*. Suppose that* Γ *and* Γ' *are connected finite* X*-regular graphs which fulfill the defining relators* R*. Put* $K = L(\Gamma, v)$ *and* $H = L(\Gamma', v')$*. Then* $\phi(K) \leq \phi(H) \leq G$*.*

Proof. By Theorem 4.5 (1), $\phi(K)$ and $\phi(H)$ are subgroups of G. By Lemma 5.2, we have $K \leq H \leq F(X)$. Since $\phi\colon F(X) \to F(X)/N = G$ is an epimorphism, $\phi(K) \leq \phi(H) \leq \phi(F(X)) = G$ holds. □

5.3 Generating systems

The next result provides a free basis for the language of an X-graph.

Recall that in a connected graph a subgraph is called a *spanning tree* if this subgraph is a tree and contains all vertices of the original graph. If T is a spanning tree, then for any two vertices u, u' of T there is a unique reduced path in T from u to u', which will be denoted $[u, u']_T$.

Lemma 5.4. (Free Basis, see [KM02, Lemma 6.1])
Let Γ *be a folded* X*-graph and let* v *be a vertex of* Γ*. Let* T *be a spanning tree of* Γ*. Let* T^+ *be the set of those edges of* Γ *which lie outside of* T*. For each* $e \in T^+$ *put* $p_e = [v, o(e)]_T e[t(e), v]_T$ *(so that* p_e *is a reduced path from* v *to* v *and its label is a freely reduced word in* $X \cup X^{-1}$*). Also for each* $e \in T^+$ *put* $[e] = \mu(p_e) = \overline{\mu(p_e)}$*. Put*

$$Y_T = \{[e] \mid e \in T^+\}.$$

Then Y_T *is a free basis for the subgroup* $H = L(\Gamma, v)$ *of* $F(X)$*.*

The free basis for the language $L(\Gamma, v_1)$ of the folded $\{a, b, c, d\}$-graph Γ shown in Figure 9 with T as a spanning tree is the set

$$Y_T = \{bab, bac, a^2, b^2da^{-1}, ba^{-1}dab^{-1}\}.$$

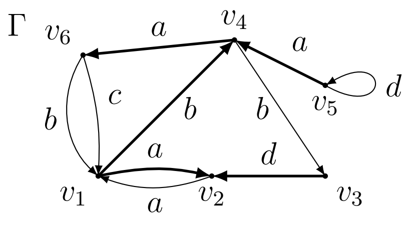

Figure 9: A folded $\{a, b, c, d\}$-graph with a spanning tree T, marked by thicker arrows.

Lemma 5.4 provides a free basis for the language of a subgroup graph. We use this to get a generating system and even a presentation for a finite index subgroup H of $G = \langle X \mid R \rangle$ from its subgroup graph $\Gamma_{X,R}(H)$. Recall that $N := \langle\!\langle R \rangle\!\rangle_{F(X)}$.

Proposition 5.5. (Generating System and Presentation)
Let $G = \langle X \mid R \rangle$ be a group with X finite and R not necessarily finite. Let Γ be a finite connected X-regular graph which fulfills the defining relators R. Let $H' = L(\Gamma, v)$ and let S be a free basis for H' which we get by Lemma 5.4. Then $\phi(S)$ generates $H = \phi(H')$. Let N' consist of the elements of N rewritten in terms of S. Then there exists an $R' \subseteq N'$ with $\langle\!\langle R' \rangle\!\rangle_{F(S)} = N'$ such that $\langle S \mid R' \rangle$ is a presentation for the finite index subgroup H of G.

Proof. The set $S \subset F(X)$ is a free basis for H'. Thus $H' = F(S)$. Since $N \leq H'$, we can rewrite N in terms of S and obtain $N' \leq F(S)$. Consequently, $H = \phi(H') = H'/N = F(S)/N'$. Moreover, there exists an $R' \subseteq N'$ such that $\langle\!\langle R' \rangle\!\rangle_{F(S)} = N'$. Hence $\langle S \mid R' \rangle$ is a presentation for H. □

It is not enough to rewrite R in terms of S to get the new relators R'. One also needs conjugates of the elements from R rewritten in terms of S. Since the subgroups we consider are all of finite index, R' can be found using the Reidemeister-Schreier method.

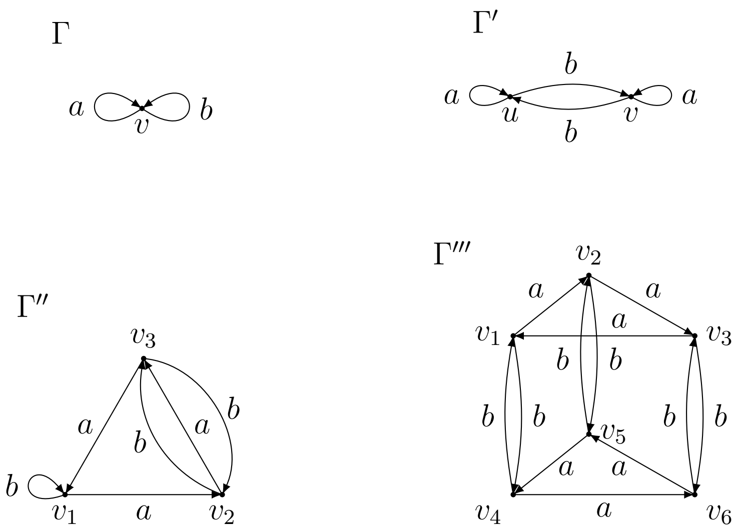

Figure 10: All $\{a, b\}$-regular graphs which fulfill the defining relators of the presentation $\langle a, b \mid a^3, b^2, (ab)^2 \rangle$ of the dihedral group D_3.

Example 5.6. We give examples for the proposition above. We consider the graphs in Figure 10, which are all $\{a, b\}$-regular graphs which fulfill the defining relators of the presentation $\langle a, b \mid a^3, b^2, (ab)^2 \rangle$ of the dihedral group D_3.

The set $\{a, b\}$ generates $L(\Gamma, v) = F(a, b)$. Hence Γ is the subgroup graph $\Gamma(D_3)$ of D_3.

The language of the graph Γ' is the same for both of its vertices. We have a free basis $\{a, bab^{-1}, b^2\}$ for the language $L(\Gamma', v)$. Since $\phi(bab^{-1}) = a^2$ and $\phi(b^2) = 1$, the graph Γ' is the subgroup graph $\Gamma(\langle a \rangle)$ of the subgroup $\langle a \rangle < D_3$.

For the graph Γ'' the languages are different for each vertex. The language of Γ'' with respect to v_1 is $F(b, aba^{-2}, a^2ba^{-1}, a^3)$. Since $\phi(aba^{-2}) = \phi(a^2ba^{-1}) = b$ and $\phi(a^3) = 1$, the subgroup graph of $\langle b \rangle < D_3$ is (Γ'', v_1). The second language for Γ'' is $L(\Gamma'', v_2) = F(a^3, ab, ba^{-1}, a^2ba^{-2})$. We have $ab = \phi(ba^{-1}) = \phi(a^2ba^{-2})$. Therefore the graph (Γ'', v_2) is the subgroup graph of $\langle ab \rangle < D_3$. The language $L(\Gamma'', v_3)$ has $\{a^3, aba^{-1}, ba^{-2}, a^2b\}$ as a free basis and ϕ gives us the subgroup $\langle a^2b \rangle < D_3$. Hence (Γ'', v_3) is the subgroup graph of $\langle a^2b \rangle < D_3$.

Since (Γ''', v_i) is isomorphic to (Γ''', v_j), the based graphs provide all the same language. The set $\{a^3, b^2, ab^2a^{-1}, a^2b^2a^{-2}, bab^{-1}a^{-2}, abab^{-1}, a^2bab^{-1}a^{-1}\}$ is a free basis for the language $L(\Gamma''', v_1)$. Since all these words are in the kernel of ϕ, the graph Γ''' is the subgroup graph $\Gamma(\{1_{D_3}\})$ of the trivial subgroup.

Remark 5.7. The example above shows that for isomorphic groups with different presentations we can have different subgroup graphs of isomorphic subgroups. Take $S_3 = \langle\, s_1, s_2 \,|\, s_1^2, s_2^2, (s_1s_2)^3 \,\rangle \cong \langle\, a, b \,|\, a^3, b^2, (ab)^2 \,\rangle = D_3$. Comparing Figures 8 and 10, we observe that the subgroup graphs of the proper subgroups are different. Since the number of vertices of a subgroup graph is the index of the associated subgroup, we have nevertheless $|V(\Gamma(H))| = |V(\Gamma(\tilde{H}))|$ for $S_3 \geq H \cong \tilde{H} \leq D_3$. Furthermore, there exists a simple connection between the graphs. Consider the map $s_1 \mapsto ab$, $s_2 \mapsto b$. We add new vertices v'' and change every edge $v \xrightarrow{s_1} v'$ to $v \xrightarrow{a} v'' \xrightarrow{b} v'$ and every edge $v \xrightarrow{s_2} v'$ to $v \xrightarrow{b} v'$ in the graphs of Figure 8. After folding these modified graphs, we get the graphs of Figure 10. Let us consider the map $ab \mapsto s_1$, $b \mapsto s_2$ and use the vertex set of a graph of Figure 10. If there exists a path $v \xrightarrow{a} v' \xrightarrow{b} v''$, then we draw an edge from v to v'' with label s_1. If there exists a path $v \xrightarrow{b} v'$, then we draw an edge labeled s_2 from v to v'. This changes the graphs of Figure 10 to the graphs of Figure 8.

5.4 Conjugate subgroups

With the next application we can detect the conjugacy class of a finite index subgroup of a finitely generated group.

Lemma 5.8. (See [KM02, Lemma 7.5])
Let Γ be a folded core graph (with respect to one of its vertices). Let v and u be two vertices of Γ and let q be a reduced path in Γ from v to u with label $g \in F(X)$. Let $H = L(\Gamma, v)$ and $K = L(\Gamma, u)$. Then $H = gKg^{-1}$.

[KM02, Proposition 7.7] and [KM02, Lemma 7.12] are formulated for subgroups of free groups and the cores of the subgroup graphs. Since the subgroup graph of a finite index subgroup is a core graph, we adapt the formulations.

Proposition 5.9. (See [KM02, Proposition 7.7])
Let H and K be finite index subgroups of $F(X)$. Then H is conjugate to K in $F(X)$ if and only if the graphs $\Gamma_X(H)$ and $\Gamma_X(K)$ are isomorphic as X-graphs.

Lemma 5.10. (See [KM02, Lemma 7.12])
Let H and K be finite index subgroups of $F(X)$. Then there is an element $g \in F(X)$ with $gKg^{-1} \leq H$ if and only if there exists a morphism of (non-based) X-graphs $\pi\colon \Gamma_X(K) \to \Gamma_X(H)$.

We extend these results to finite index subgroups of finitely generated groups.

Lemma 5.11.
Let $G = \langle X \mid R \rangle$ be a group with X finite and R not necessarily finite. Let Γ be a finite X-regular graph which fulfills the relators R. Let v and u be two vertices of Γ and let p be a reduced path from v to u with label $g' \in F(X)$. Let $H = \phi(L(\Gamma, v))$ and $K = \phi(L(\Gamma, u))$. Then $H = gKg^{-1}$ for $g = \phi(g') \in G$.

Proof. Let $H' = L(\Gamma, v)$ and $K' = L(\Gamma, u)$. By Lemma 5.8, $H' = g'K'g'^{-1}$ for $g' \in F(X)$ as above. Since ϕ is a homomorphism, $H = \phi(H') = gKg^{-1}$ for $g = \phi(g') \in G$. □

Therefore the subgroup $H = \phi(L(\Gamma_{X,R}(H), 1_H))$ is conjugate to the subgroup $\phi(L(\Gamma_{X,R}(H), v))$ in G for all $v \in V(\Gamma_{X,R}(H))$.

Proposition 5.12. (Conjugate Subgroups)
Let H and K be subgroups of finite index in the group $G = \langle X \mid R \rangle$, where X is finite and R is not necessarily finite. Then H is conjugate to K in G if and only if the subgroup graphs $\Gamma_{X,R}(H)$ and $\Gamma_{X,R}(K)$ are isomorphic as X-graphs.

Proof. Let $H' = L(\Gamma_{X,R}(H), 1_H)$ and $K' = L(\Gamma_{X,R}(K), 1_K)$. Then H' and K' are finite index subgroups of $F(X)$ and $\Gamma_{X,R}(H) = \Gamma_X(H')$ and $\Gamma_{X,R}(K) = \Gamma_X(K')$. Since H and K are of finite index and conjugate, H' and K' are conjugate. Lemma 5.11 and Proposition 5.9 complete the proof. □

Lemma 5.13.
Let H and K be finite index subgroups of the group $G = \langle X \mid R \rangle$, where X is finite and R is not necessarily finite. Then there is $g \in G$ with $gKg^{-1} \leq H$ if and only if there exists a morphism of (non-based) X-graphs $\pi\colon \Gamma_{X,R}(K) \to \Gamma_{X,R}(H)$.

Proof. This follows from Lemma 5.10 and the proof of Proposition 5.9. □

5.5 Normal subgroups and normalizers

Proposition 5.12 states that for a finite index subgroup H of a group G the set $\{\phi(L(\Gamma(H), v)) \mid v \in V(\Gamma(H))\}$ is the conjugacy class of H. Thus H has at most $|V(\Gamma(H))|$ conjugate subgroups. This leads to the next result.

Theorem 5.14. (Normal Subgroups)
Let H be a finite index subgroup of the group $G = \langle X \mid R \rangle$, where X is finite and R is not necessarily finite. Then H is normal in G if and only if the based X-graphs $(\Gamma_{X,R}(H), 1_H)$ and $(\Gamma_{X,R}(H), v)$ are isomorphic for all $v \in V(\Gamma_{X,R}(H))$.

Proof. $H = \phi(L(\Gamma_{X,R}(H), 1_H))$ is conjugate to $\phi(L(\Gamma_{X,R}(H), v))$. The subgroup H is normal if and only if H is conjugate only to itself. This is equivalent to $\phi(L(\Gamma_{X,R}(H), v)) = H$ for all $v \in V(\Gamma_{X,R}(H))$. Hence $(\Gamma_{X,R}(H), 1_H)$ and $(\Gamma_{X,R}(H), v)$ are isomorphic. □

With the subgroup graph we can detect the normalizer of a subgroup. The normalizer of a subgroup H in a group G is the subgroup

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

Theorem 5.15. (Normalizer)
Let $G = \langle X \mid R \rangle$ be a group with X finite and R not necessarily finite. Let H be a finite index subgroup of G. Let p_v be the reduced path in $\Gamma_{X,R}(H)$ from 1_H to v with label $\mu(p_v) = g$. Then $\phi(g) = g_v \in N_G(H)$ if and only if $(\Gamma_{X,R}(H), 1_H)$ and $(\Gamma_{X,R}(H), v)$ are isomorphic as based X-graphs. Furthermore, let V be the set of vertices of $\Gamma_{X,R}(H)$ with $(\Gamma_{X,R}(H), 1_H)$ isomorphic to $(\Gamma_{X,R}(H), v)$ as based X-graphs. Then

$$N_G(H) = \bigcup_{v \in V} Hg_v.$$

Proof. Let $g_v \in G$ and let p_v be the reduced path with label g_v and origin 1_H in $\Gamma(H) := \Gamma_{X,R}(H)$. Let v be the terminus of p_v. By Lemma 5.11, we have $H = g_v K g_v^{-1}$ for $K = \phi(L(\Gamma(H), v)) \leq G$. If $g_v \in N_G(H)$, then $K = g_v^{-1} H g_v = H$. Therefore $(\Gamma(H), 1_H)$ and $(\Gamma(H), v)$ are isomorphic as based X-graphs.

Let $(\Gamma(H), 1_H)$ and $(\Gamma(H), v)$ be isomorphic as based X-graphs. Then the subgroups $H = \phi(L(\Gamma(H), 1_H))$ and $K = \phi(L(\Gamma(H), v))$ are equal. Let p_v be the reduced path from 1_H to v with label g_v. Then $H = g_v K g_v^{-1} = g_v H g_v^{-1}$. Hence $g_v \in N_G(H)$.

Let $V := \{v \in V(\Gamma(H)) \mid (\Gamma(H), 1_H) \cong (\Gamma(H), v)\}$. If $g_v \in N_G(H)$, then $Hg_v \subseteq N_G(H)$. Let $g \in N_G(H)$ and let p be the reduced path with label g, origin 1_H and terminus v'. Since $g \in N_G(H)$, the based graphs $(\Gamma(H), 1_H)$ and $(\Gamma(H), v')$ are isomorphic. Therefore $v' \in V$. □

Theorem 5.15 shows that if there is no symmetry in the subgroup graph $\Gamma(H)$ of a subgroup $H < G$ (that is $(\Gamma(H), 1_H) \ncong (\Gamma(H), v)$ for all $v \neq 1_H$), then $N_G(H) = H$.

5.6 Index and intersection of subgroups

We introduce the subgroup graph of the intersection of two finite index subgroups. In the following subsections we use $\Gamma(H)$ for the subgroup graph of the subgroup H for both $H \leq F(X)$ and $H \leq G$. This is less precise but clearer to read.

Definition 5.16. (Product Graph, see [KM02, Definition 9.1])
Let Γ and Γ' be X-graphs. We define the *product graph* $\Gamma \times \Gamma'$ as follows. The vertex set of $\Gamma \times \Gamma'$ is the set $V(\Gamma) \times V(\Gamma')$. For a pair of vertices (u, v), $(u', v') \in V(\Gamma \times \Gamma')$ (such that $u, u' \in V(\Gamma)$ and $v, v' \in V(\Gamma')$) and a letter $x \in X$ we introduce an edge, labeled x, with origin (u, v) and terminus (u', v'), provided that there is an edge, labeled x, from u to u' in Γ and there is an edge, labeled x, from v to v' in Γ'.

Thus $\Gamma \times \Gamma'$ is an X-graph. We denote a vertex (u, v) of the product graph $\Gamma \times \Gamma'$ by $u \times v$.

For an example of a product graph see Figure 11. The graph in the second row is a product graph of the two graphs in the first row.

Lemma 5.17. (See [KM02, Lemma 9.2])
Suppose Γ and Γ' are folded X-graphs. Then $\Gamma \times \Gamma'$ is also a folded X-graph.

Proposition 5.18. (See [KM02, Proposition 9.4])
Let H and K be two subgroups of $F(X)$. Let $\Gamma(H) \times_1 \Gamma(K)$ be the connected component of the product graph $\Gamma(H)\times\Gamma(K)$ containing $1_H \times 1_K$ and Δ be the core of $\Gamma(H)\times_1\Gamma(K)$ with respect to $1_H \times 1_K$. Then $(\Gamma(H\cap K), 1_{H\cap K}) = (\Delta, 1_H \times 1_K)$.

To translate Proposition 5.18 we need the following two lemmas.

Lemma 5.19.
Let Γ and Γ' be two finite connected X-regular graphs. Then the product graph $\Gamma \times \Gamma'$ is a finite X-regular graph.

Proof. Let $v \times v'$ be a vertex in $\Gamma \times \Gamma'$ and let $x \in X \cup X^{-1}$. Since Γ and Γ' are X-regular, there exists exactly one edge e with label x and origin v in Γ and exactly one edge e' with label x and origin v' in Γ'. By Definition 3.15, there exists exactly one edge in $\Gamma \times \Gamma'$ with label x and origin $v \times v'$. Hence $\Gamma \times \Gamma'$ is X-regular. Since Γ and Γ' are finite, the product graph is finite. □

Lemma 5.20.
Let $G = \langle X \mid R \rangle$ be a group with X finite and R not necessarily finite. Let Γ and Γ' be two finite X-regular graphs which fulfill the defining relators R. Then $\Gamma \times \Gamma'$ fulfills the defining relators R.

Proof. By Lemma 5.19, $\Gamma \times \Gamma'$ is a finite X-regular graph. Let $v \times v'$ be a vertex in $\Gamma\times\Gamma'$ and $r \in R$. Then there exists exactly one reduced path p_r with $\mu(p_r) = r$ and $o(p_r) = v \times v'$. Since Γ and Γ' fulfill the defining relators R, the reduced path p in Γ with $\mu(p) = r$ and $o(p) = v$ has terminus v and the reduced path p' in Γ' with $\mu(p') = r$ and $o(p_r) = v'$ has terminus v'. Therefore $t(p_r) = v \times v'$ in $\Gamma \times \Gamma'$. □

Proposition 5.21. (Intersection)
Let H and K be finite index subgroups of the group $G = \langle X \mid R \rangle$, where X is finite and R is not necessarily finite. Let $\Gamma(H) \times_1 \Gamma(K)$ be the connected component of the product graph $\Gamma(H)\times\Gamma(K)$ containing $1_H \times 1_K$. Then $(\Gamma(H)\times_1\Gamma(K), 1_H \times 1_K)$ is the subgroup graph of $H \cap K < G$.

Proof. By Theorem 4.5, the graphs $\Gamma(H)$ and $\Gamma(K)$ are finite, connected, X-regular and fulfill the defining relators R. By Lemmas 5.19 and 5.20, the product graph $\Gamma(H) \times \Gamma(K)$ is finite, X-regular and fulfills the defining relators R. Thus $\Delta = Core(\Gamma(H) \times_1 \Gamma(K), 1_H \times 1_K) = \Gamma(H) \times_1 \Gamma(K)$ is X-regular and fulfills the defining relators R. Recall that $\Gamma(H) = \Gamma(H')$ and $\Gamma(K) = \Gamma(K')$ for $H' = L(\Gamma(H), 1_H)$ and $K' = L(\Gamma(K), 1_K)$. By Proposition 5.18, we know that $(\Delta, 1_H\times 1_K)$ is the subgroup graph of $H'\cap K'$. Thus $L(\Delta, 1_H\times 1_K) = H'\cap K'$. Hence $\phi(L(\Delta, 1_H \times 1_K)) = \phi(H' \cap K')$. Since ϕ is a homomorphism and $\ker\phi \leq H' \cap K'$, we have $\phi(H' \cap K') = \phi(H') \cap \phi(K') = H \cap K$. □

Since the number of vertices of a subgroup graph of a finite index subgroup is the index of the subgroup, we get the following known corollary.

Corollary 5.22.
Let H and K be two finite index subgroups of the finitely generated group G. Then $[G : H \cap K] \leq [G : H][G : K]$.

Proposition 5.23. (Intersection of Cosets)
Let H and K be finite index subgroups of the group $G = \langle\, X \mid R \,\rangle$, where X is finite and R is not necessarily finite. If Hg_v is a vertex $v \in V(\Gamma(H))$ and $Kg_{v'}$ a vertex $v' \in V(\Gamma(K))$, then $v \times v'$ is a vertex in $\Gamma(H) \times_1 \Gamma(K)$ if and only if the intersection of the cosets Hg_v and $Kg_{v'}$ is not empty. Furthermore, if $v \times v'$ is in $\Gamma(H) \times_1 \Gamma(K)$ and g' is the label of a reduced path from $1_H \times 1_K$ to $v \times v'$, then $Hg_v \cap Kg_{v'} = (H \cap K)g_{v \times v'}$ with $\phi(g') = g_{v\times v'} \in G$.

Proof. Let $v \times v'$ be a vertex in $\Gamma(H) \times_1 \Gamma(K)$. Let p be the reduced path from $1_H \times 1_K$ to $v \times v'$ and $\mu(p) = g$. By definition, there exists a reduced path p' from 1_H to v in $\Gamma(H)$ and a reduced path p'' from 1_K to v' in $\Gamma(K)$ with $\mu(p') = \mu(p'') = g$. Therefore $\phi(g)$ is in $Hg_v \cap Kg_{v'}$. Hence $(H \cap K)g_{v\times v'} \subseteq Hg_v \cap Kg_{v'}$.

Let $g \in Hg_v \cap Kg_{v'}$. Hence there is a reduced path p with $\mu(p) = g$ from 1_H to v in $\Gamma(H)$ and a reduced path p' with label g from 1_K to v' in $\Gamma(K)$. Thus the vertex $v \times v'$ is in $V(\Gamma(H) \times_1 \Gamma(K))$ and the reduced path with label g is from $1_H \times 1_K$ to $v \times v'$. Therefore $g \in (H \cap K)g_{v\times v'}$. Consequently, $(H \cap K)g_{v\times v'} = Hg_v \cap Kg_{v'}$. □

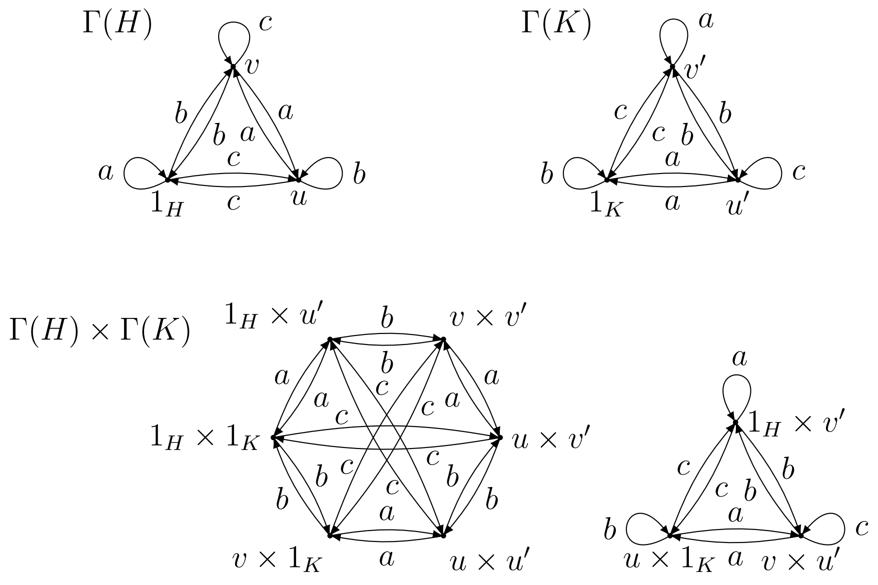

Figure 11: Two subgroup graphs $\Gamma(H)$, $\Gamma(K)$ at the top and their disconnected product graph $\Gamma(H) \times \Gamma(K)$ at the bottom.

Example 5.24. The graph $(\Gamma(H), 1_H)$ of Figure 11 is the subgroup graph of the subgroup $H = \langle a, cbc, cab\rangle$ and $(\Gamma(K), 1_K)$ is the subgroup graph of the subgroup $K = \langle b, aca, abc\rangle$ of the group $\Delta(3, 3, 3) = \langle\, a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3 \,\rangle$ (Coxeter group of type $\widetilde{A}_2$). The connected component of $1_H \times 1_K$ is the subgroup graph of the intersection $H \cap K$. Hence $H \cap K$ is a normal subgroup of index 6 in $\Delta(3, 3, 3)$. From the subgroup graphs $\Gamma(H)$ and $\Gamma(K)$ we get the cosets $H = 1_H$, $Hb = v$, $Hc = u$ of the subgroup H and the cosets $K = 1_K$, $Kc = v'$, $Ka = u'$ of the subgroup K. Using the product graph $\Gamma(H) \times \Gamma(K)$, we get the following intersection of the cosets: $H \cap K$; $H \cap Ka = (H \cap K)a$; $Hb \cap K = (H \cap K)b$ and

$Hc \cap Kc = (H \cap K)c$. Furthermore, there exist paths $1_H \times 1_K \xrightarrow{ab} v \times v'$ and $1_H \times 1_K \xrightarrow{ba} u \times u'$, therefore $Hb \cap Kc = (H \cap K)ab$ and $Hc \cap Ka = (H \cap K)ba$. Since $1_H \times v'$, $u \times 1_K$ and $v \times u'$ are not in $\Gamma(H) \times_1 \Gamma(K)$, the intersections $H \cap Kc$, $Hc \cap K$ and $Hb \cap Ka$ are empty.

In the following we study the index of an intersection of subgroups. Some of the following propositions are known. We prove them using subgroup graphs.

Let $deg_x(v)$ denote the number of incoming and outgoing edges with label x. Since an loop is incoming and outgoing, we count it as 2. If an X-graph is folded, then $deg_x(v) \leq 2$ and if it is X-regular, then $deg_x(v) = 2$ for all $x \in X$.

Lemma 5.25.
Let Γ be a finite connected X-regular graph and let Γ' be a finite folded connected X-graph. Let v' be a vertex of Γ' and $x \in X$. Then $deg_x(v') = deg_x(v \times v')$ for all $v \in V(\Gamma)$.

Proof. Since Γ is X-regular, for every vertex $v \in V(\Gamma)$ and for each $x \in X$ there is an edge with label x and origin v and an edge with label x and terminus v. Thus there exists an edge with label x and origin or terminus $v \times v'$ in $\Gamma \times \Gamma'$ if and only if there exists an edge with label x and origin or terminus v', respectively, in Γ'. □

In other words, if there is an edge e from u' to v' with label x in Γ', then there is an edge from $u \times u'$ to $v \times v'$ with label x for all $u \in V(\Gamma)$. Then $u \times u' = v \times v'$ if and only if $u = v$ and $u' = v'$. Hence a loop can become a non-loop, but not conversely.

Lemma 5.26.
Let Γ be a finite connected X-regular graph with $|V(\Gamma)| = m$. Let T be a spanning tree of a finite connected folded X-graph Γ' with $|V(\Gamma')| = n$ vertices. Then the product graph $\Gamma \times T$ is isomorphic to a folded X-graph with m connected components and each component is isomorphic to T.

Proof. Let $w(u)$ be the terminus of the reduced path with label w and origin u in Γ. Let $v_1, v_2, ..., v_n$ be the vertices of T. Let p_i be the reduced path from v_1 to v_i in T. Since T is a tree and p_i is reduced, p_i is unique. Let w_i be the label of the path p_i. Since Γ is X-regular, there exists a reduced path with label w_i from $u \times v_1$ to $w_i(u) \times v_i$ in $\Gamma \times T$. Therefore the vertices $u \times v_1$, $w_2(u) \times v_2$, ..., $w_n(u) \times v_n$ are in $\Gamma \times_{u \times v_1} T$. Suppose there is another vertex $u' \times v_i$ in $\Gamma \times_{u \times v_1} T$ with $u' \neq w_i(u)$. Then there is a reduced path from $w_i(u) \times v_i$ to $u' \times v_i$. Since each reduced path from v_i to v_i in T is the trivial path, $u' = w_i(u)$. Consequently, the map $\Gamma \times_{u \times v_1} T \to T$ with $w_i(u) \times v_i \mapsto v_i$ is bijective on the vertex set. Thus $\Gamma \times T$ has m connected components.

Suppose the vertices $w_i(u) \times v_i$ and $w_j(u) \times v_j$ are adjacent in $\Gamma \times_{u \times v_1} T$ for $i \neq j$. Since T is a tree, there is only one edge connecting the vertices v_i and v_j. By Lemma 5.25, there is only one edge connecting $w_i(u) \times v_i$ and $w_j(u) \times v_j$. This edge must have the same label and orientation as the edge connecting v_i and v_j. Therefore $\Gamma \times_{u \times v_1} T$ is isomorphic to T for each $u \in V(\Gamma)$. □

Lemma 5.27.
Let $G = \langle X \mid R \rangle$ be a finitely generated group and $H, K \in \mathscr{H}_{\text{fi}}(G)$. Then $[G:H]$ and $[G:K]$ divide $[G:H \cap K]$.

Proof. Since H and K are finite index subgroups, there exist subgroup graphs $(\Gamma(H), 1_H)$ and $(\Gamma(K), 1_K)$ with $[G:H] = m$ and $[G:K] = n$ vertices. Let T be a spanning tree of $\Gamma(K)$. By Lemma 5.26, $\Gamma(H) \times \Gamma(K)$ contains m copies of T. Since $\Gamma(K)$ is X-regular, it has more edges than T. These edges may connect the copies of T. Thus $\Gamma(H \cap K) = \Gamma(H) \times_1 \Gamma(K)$ has tn vertices, for some $1 \leq t \leq m$. Analogously we prove $[G : H \cap K] = sm$ for some $1 \leq s \leq n$. □

Lemma 5.28.
Let $G = \langle X \mid R \rangle$ be a finitely generated group and $H, K \in \mathscr{H}_{\text{fi}}(G)$. Then $[G : H \cap K] = [G:H][G:K]$ if and only if $G = HK$.

Proof. Let $(\Gamma(H), 1_H)$ be the subgroup graph of H and $(\Gamma(K), 1_K)$ the subgroup graph of K. We know that the vertex v in $\Gamma(H)$ represents the coset Hg_v, where $g_v = \mu(p_v)$ with p_v being a reduced path from 1_H to v. Moreover, $Hg_v \cap Kg_w \neq \varnothing$ if and only if there exists a path from $1_H \times 1_K$ to $v \times w$ in $\Gamma(H) \times \Gamma(K)$. We know that $G = HK$ if and only if $Hg \cap K \neq \varnothing$ for all $g \in G$. Which is equivalent to the statement that for all $v \in V(\Gamma(H))$ there exists a path p_v from $1_H \times 1_K$ to $v \times 1_K$. Let T be a spanning tree of $\Gamma(K)$. By Lemma 5.26, the graph $\Gamma(H) \times \Gamma(K)$ contains $[G : H]$ copies of the tree T, one for each vertex $v \times 1_K$ with $v \in V(\Gamma(H))$. Thus if $G = HK$, then this vertices $v \times 1_K$ are connected and thus $\Gamma(H) \times \Gamma(K) = \Gamma(H \cap K)$, which is equivalent to $[G : H \cap K] = [G : H][G : K]$. Conversely, if $\Gamma(H) \times \Gamma(K)$ has only one connected component, then $G = HK$. □

5.7 Malnormal subgroups

A subgroup H of a group G is called *malnormal* if for all $g \in G \setminus H$

$$gHg^{-1} \cap H = \{1_G\}.$$

The groups G and $\{1_G\}$ are always malnormal.

The next two propositions are needed for the theorem for malnormal subgroups.

Proposition 5.29. (See [KM02, Proposition 9.7])
Let H and K be subgroups of $F(X)$. Let $g \in F(X)$ be such that the double cosets KgH and KH are distinct. Suppose that $gHg^{-1} \cap K \neq \{1\}$. Then there is a vertex $v \times u$ in $\Gamma(H) \times \Gamma(K)$ which does not belong to the connected component of $1_H \times 1_K$ such that the subgroup $L(\Gamma(H) \times \Gamma(K), v \times u)$ is conjugate to $gHg^{-1} \cap K$ in $F(X)$.

Proposition 5.30. (See [KM02, Proposition 9.8])
Let H and K be subgroups of $F(X)$. Then for any vertex $v \times u$ of $\Gamma(H) \times \Gamma(K)$ the subgroup $L(\Gamma(H) \times \Gamma(K), v \times u)$ is conjugate to a subgroup of the form $gHg^{-1} \cap K$ for some $g \in F(X)$. Moreover, if $v \times u$ does not belong to the connected component of $1_H \times 1_K$, then the element g can be chosen such that $KgH \neq KH$.

Theorem 5.31. (See [KM02, Theorem 9.10])
Let $H \leq F(X)$ be a subgroup of $F(X)$. Then H is malnormal in $F(X)$ if and only if every component of $\Gamma(H) \times \Gamma(H)$, which does not contain $1_H \times 1_H$, is a tree.

The intersection of two finite index subgroups has finite index. Hence no proper finite index subgroup of an infinite finitely generated group is malnormal. Therefore the next application restricts to malnormal subgroups of finite groups.

Theorem 5.32. (Malnormal Subgroups)
Let H be a subgroup of a finite group $G = \langle X \mid R \rangle$. The subgroup H is malnormal in G if and only if $L(\Gamma(H) \times \Gamma(H), u \times v) = N$ for all vertices $u \times v$ not in the connected component of $1_H \times 1_H$.

Proof. Let $u \times v$ be not in the connected component of $1_H \times 1_H$. Assume $L(\Gamma(H) \times \Gamma(H), u \times v) \neq N$. Since $\Gamma(H)$ is the subgroup graph of H, it fulfills the defining relators R. Thus every connected component of the product graph $\Gamma(H) \times \Gamma(H)$ fulfills the defining relators R by Lemma 5.20. From Proposition 4.4 it follows that $N \leq L(\Gamma(H) \times \Gamma(H), u \times v)$. Thus $\phi(L(\Gamma(H) \times \Gamma(H), u \times v)) \neq \{1_G\}$. Let $H' := \phi^{-1}(H)$. Then $\Gamma(H) = \Gamma(H')$. Proposition 5.30 now shows that $L(\Gamma(H) \times \Gamma(H), u \times v)$ is conjugate to $g'H'g'^{-1} \cap H'$ for some $g' \in F(X) \setminus H'$. Since $\ker \phi \leq H'$, we have $\phi(g'H'g'^{-1} \cap H') = gHg^{-1} \cap H$ for $g = \phi(g')$. Therefore $\phi(L(\Gamma(H) \times \Gamma(H), u \times v))$ is conjugate to $gHg^{-1} \cap H$ for $g \in G \setminus H$. But $\phi(L(\Gamma(H) \times \Gamma(H), u \times v)) \neq \{1_G\}$. Thus $gHg^{-1} \cap H \neq \{1_G\}$ and H is not malnormal.

Let $L(\Gamma(H) \times \Gamma(H), u \times v) = N$ for all vertices $u \times v$ not in the connected component of $1_H \times 1_H$. Let $g \in G \setminus H$, then every $g' \in F(X)$ with $\phi(g') = g$ is not in H'. Thus we have $H'g'H' \neq H'H'$. Assume $g'H'g'^{-1} \cap H' \neq \{1_{F(X)}\}$. From Proposition 5.29 it follows that there exists a vertex $u \times v$ not in the connected component of $1_H \times 1_H$ such that $L(\Gamma(H) \times \Gamma(H), u \times v)$ is conjugate to $g'H'g'^{-1} \cap H'$. Since $L(\Gamma(H) \times \Gamma(H), u \times v) = N$, we have $g'H'g'^{-1} \cap H' = N$. Thus $gHg^{-1} \cap H = \{1_G\}$. Assume $g'H'g'^{-1} \cap H' = \{1_{F(X)}\}$, then $gHg^{-1} \cap H = \{1_G\}$. Therefore H is malnormal. □

Figure 12 shows that $H = \langle s_1 \rangle$ is malnormal in $S_3 = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$, since the right connected component of $\Gamma(H) \times \Gamma(H)$ is isomorphic to $\Gamma(1_{S_3})$.

Corollary 5.33.
Let G be a finite group and $H \leq G$ a subgroup of index n. If H is malnormal, then $|G|$ divides $n^2 - n$.

Proof. Suppose H is malnormal and has index n in G. Therefore the subgroup graph $\Gamma(H)$ has n vertices. Thus $\Gamma(H) \times \Gamma(H)$ has n^2 vertices. By Theorem 5.32, $\Gamma(H) \times_1 \Gamma(H) \cong \Gamma(H)$ and the other connected components are isomorphic to $\Gamma(1_G)$. The subgroup graph $\Gamma(1_G)$ has $|G|$ vertices. Hence $|G|$ divides $n^2 - n$. □

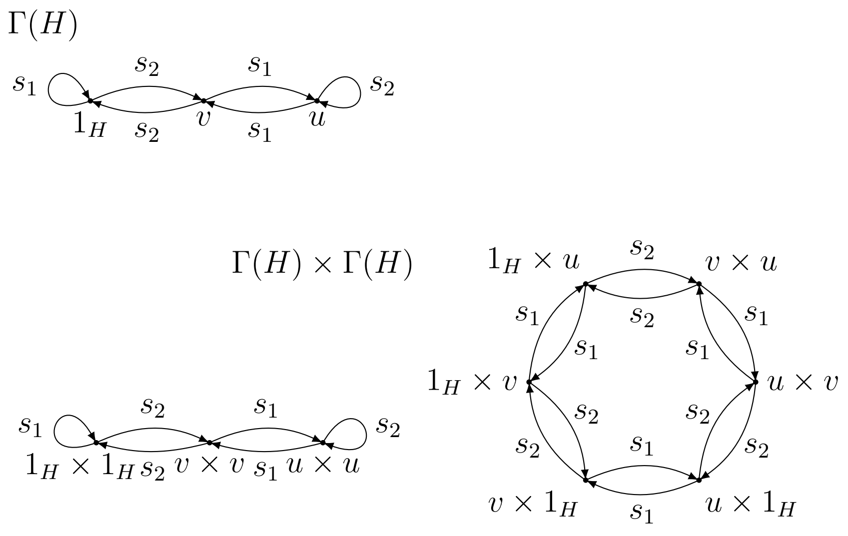

Figure 12: The subgroup graph $\Gamma(H)$ at the top and the disconnected product graph $\Gamma(H) \times \Gamma(H)$ of the subgroup $H = \langle s_1 \rangle$ of $S_3 = \langle\, s_1, s_2 \,|\, s_1^2, s_2^2, (s_1 s_2)^3 \,\rangle$ at the bottom.

Part III
(Finite index) coset poset of finitely generated infinite groups

In the last part of this thesis we study the finite index coset poset and the coset poset of finitely generated infinite groups. As we have stated in Section 2 the contractibility of the coset poset $\mathscr{C}(G)$ is known for finite groups and for infinitely generated groups. D.A. Ramras proved in [Ram05] that $\mathscr{C}(G)$ is contractible for all infinitely generated groups G and J. Shareshian and R. Woodroofe proved in [SW16] that $\mathscr{C}(G)$ is not contractible for all finite groups G. Moreover, the connectivity of the coset poset of finitely generated infinite groups is only known for two families of groups, Tarski monster groups and groups whose non-trivial proper subgroups are infinite cyclic. Therefore we are interested in the homotopy type of the coset poset of finitely generated infinite groups. To study the coset poset $\mathscr{C}(G)$ of a finitely generated infinite group G, we first study a subset. This subset is the finite index coset poset $\mathscr{C}_{\text{fi}}(G)$, the set of all proper finite index subgroups and their cosets. Since finite index subgroups are important in finitely generated groups, the finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is interesting by itself. We study the homotopy type of the coset poset $\mathscr{C}(G)$ and the finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ studying the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$, respectively. For the finite index case we use our theory of subgroup graphs. Therefore this part studies nerve complexes and shows the usage of our subgroup graph theory.

Section 6 generalizes and extends the results of the author's article [Wel17, Sections 5 and 6]. In the article we only used subgroup graphs, therefore we proved the results in a slightly different way. In this thesis we have stronger results, since we use, in addition to the subgroup graphs, our Cone Argument.

In Sections 6 and 7 we study the homotopy type of the finite index coset poset. In Section 6 we prove that $\mathscr{C}_{\text{fi}}(G)$ is contractible for various classes of finitely generated infinite groups. In contrast, Section 7 proves that there exist finitely generated infinite groups, apart from those where $\mathscr{C}_{\text{fi}}(G)$ is empty, for which the finite index coset poset is non-contractible. Moreover, we study the importance of the maximal subgroups for the homotopy type of the finite index coset poset. This raises new questions. In Section 8 we study the homotopy type of the coset poset $\mathscr{C}(G)$ of finitely generated infinite groups. We prove that there exist examples of finitely generated infinite groups both for contractible and for non-contractible coset posets. Therefore we obtained two non-trivial homotopy invariants for finitely generated infinite groups. Moreover, we prove that there exist finitely generated infinite groups whose coset poset is non-contractible and has the homotopy type of a finite or infinite simplicial complex. This leads to some conjectures about the importance of the cardinality of the two sets of maximal subgroups $\mathscr{H}_{\max}$ and $\mathscr{H}_{\max,\text{fi}}$. Finally, we translate our results of the connectivity of the nerve complexes to those of higher generation of subgroups.

6 Contractible finite index coset poset

In this section we prove that there exist finitely generated infinite groups whose finite index coset poset is contractible. Moreover, our study provides information about the connectivity of some special nerve complexes.

In Section 6.1 we start with the definition of the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$, see Definition 6.1. Afterwards we prove that the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ and the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ are homotopy equivalent, see Proposition 6.2. Therefore it suffices to study the homotopy type of the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. We study it using the Cone Argument 6.4, which states the following. For any finite subcomplex U of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$, there exists a cone $U * \{K\}$ in the nerve complex if and only if the set $\mathscr{H}_{coH,\mathrm{max},\mathrm{fi}}(G)$ is infinite for all finite index subgroups $H \in \mathscr{H}_{\mathrm{fi}}(G)$. Thus, if the latter holds, the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ is contractible. Afterwards we prove that $\mathscr{H}_{coH,\mathrm{max},\mathrm{fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\mathrm{fi}}(G)$ if one of the following sets is infinite: the set of all normal maximal subgroups $\mathscr{H}_{\mathrm{nor,max}}(G)$, the set of all subgroups of prime index $\mathscr{H}_{\mathbb{P}}(G)$, or the set of all subgroups of bounded prime power index $\mathscr{H}_{\mathbb{P}^n}(G)$, see Theorems 6.6 and 6.9 and Corollary 6.12, respectively.

Section 6.2 states examples of groups whose finite index coset poset is contractible. Using subgroup graphs, we study groups with $\mathscr{H}_{\mathrm{nor,max}}$, $\mathscr{H}_{\mathbb{P}}$, or $\mathscr{H}_{\mathbb{P}^n}$ being infinite in Section 6.2.1, Section 6.2.2, or Section 6.2.3, respectively. This leads to the results that the finite index coset poset is contractible for many important classes of finitely generated infinite groups as the free groups, the free abelian groups, the right angled Coxeter groups, the Artin groups, all pure braid groups, Fuchsian groups of genus $g \geq 2$, Baumslag-Solitar groups, Thompson's group F, HNN extensions, many infinite triangle groups, or infinite special linear groups $\mathrm{SL}(n, \mathbb{Z})$.

In Section 6.3 we consider more general conditions for a contractible finite index coset poset. Therefore we consider free, direct, semidirect, and amalgamated products. We prove Corollary 6.25, which states that the finite index coset poset of the free product of finitely many finitely generated groups is contractible if at least two of them have proper finite index subgroups. The groups need not be infinite. Moreover, we prove that if $\mathscr{H}_{\mathrm{nor,max}}(G_1)$, $\mathscr{H}_{\mathbb{P}}(G_1)$, or $\mathscr{H}_{\mathbb{P}^n}(G_1)$ is infinite, then $\mathscr{H}_{\mathrm{nor,max}}(G)$, $\mathscr{H}_{\mathbb{P}}(G)$, or $\mathscr{H}_{\mathbb{P}^n}(G)$, respectively, is infinite for G being a free product, direct product, or a semi direct product of finitely generated groups $G_1, ..., G_n$, see Corollary 6.27. Finally, we prove Theorem 6.31, which states that for a finite index subgroup H of G the finite index coset poset $\mathscr{H}_{\mathrm{fi}}(H)$ inherits the contractibility from $\mathscr{C}_{\mathrm{fi}}(G)$ if $\mathscr{H}_{coK,\mathrm{max},\mathrm{fi}}(G)$ is infinite for each $K \in \mathscr{H}_{\mathrm{fi}}(G)$.

6.1 Method

An infinite simplicial complex K is contractible if and only if every finite subcomplex U is contractible in K. We prove that every finite subcomplex U is contractible in $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ in the following way. For every U, there exists a vertex $v \notin V(U)$ such that the join $U * \{v\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. The join of a space X with a one-point space is a cone $C(X)$ of X. Thus the join $U * \{v\}$ is a cone $C(U)$ of U. We prove the existence of such cones by proving the existence of infinite sets of special maximal subgroups.

Definition 6.1. (Finite index coset poset $\mathscr{C}_{\text{fi}}(G)$)
Let G be a finitely generated group and

$$\mathscr{H}_{\text{fi}}(G) := \{H < G \mid 1 < [G : H] < \infty\}. \tag{6.1.1}$$

The *finite index coset poset* $\mathscr{C}_{\text{fi}}(G)$ is the set

$$\mathscr{C}_{\text{fi}}(G) := \{Hg \mid H \in \mathscr{H}_{\text{fi}}(G), g \in G\}, \tag{6.1.2}$$

ordered by inclusion.

If G is a finite group, then $\mathscr{H}_{\text{fi}}(G) = \mathscr{H}_{\mathscr{C}}(G)$ and $\mathscr{C}_{\text{fi}}(G) = \mathscr{C}(G)$. Thus for finite groups there is no difference between the finite index coset poset and the coset poset.

Recall that in the notation of the nerve complex $\mathcal{NC}(G, \mathscr{H})$, Definition 2.9, the set $\mathscr{H}$ is always a set of subgroups of G and the vertex set of $\mathcal{NC}(G, \mathscr{H})$ is the set of cosets of the subgroups of $\mathscr{H}$ in G. Therefore we write $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ instead of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}}(G))$. In fact, if G is understood we often write $\mathscr{H}_{\text{fi}}$ instead of $\mathscr{H}_{\text{fi}}(G)$.

Proposition 6.2.
Let G be a finitely generated group. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ is homotopy equivalent to the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$.

Proof. Since G is a finitely generated group, $\mathscr{H}_{\text{fi}}(G)$ and $\mathscr{C}_{\text{fi}}(G)$ are closed under intersection. Consequently, $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ and $\Delta\mathscr{C}_{\text{fi}}(G)$ are homotopy equivalent, by Theorem 2.10. □

As for the coset poset of finite groups, we study the homotopy type of the nerve complex instead of the order complex.

Lemma 6.3.
Let U be a finite subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. Suppose that there exists a subgroup $K \in \mathscr{H}_{\text{fi}}(G)$ such that, for all vertices $Hg \in U$, we have $HK = G$. Then the join $U * \{K\}$ is a contractible subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$.

Proof. It is easy to see that $G = HK$ is equivalent to $G = \bigcup_{k \in K} Hk$, which is equivalent to $Hg \cap K \neq \varnothing$ for all $g \in G$. Let U be a finite subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. Let $\sigma = \{H_0g_0, ..., H_ng_n\}$ be an n-simplex in U, then $\bigcap \sigma \neq \varnothing$. By the assumption, $G = H_iK$. Thus $H_ig_i \cap K \neq \varnothing$ for all $i = 0, ..., n$. Consequently, $H_0g_0 \cap ... \cap H_ng_n \cap K \neq \varnothing$. Therefore, $\sigma \cup \{K\}$ is a simplex in $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ for all $\sigma \in U$. Hence the join $U * \{K\}$ is a contractible subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. □

For a finitely generated group G and an arbitrary subgroup H we define the following subsets of $\mathscr{H}_{\text{fi}}(G)$

$$\mathscr{H}_{\max,\text{fi}}(G) := \{M \in \mathscr{H}_{\text{fi}}(G) \mid M \text{ maximal subgroup of } G\} \tag{6.1.3}$$

and

$$\mathscr{H}_{coH,\max,\mathrm{fi}}(G) := \{M \in \mathscr{H}_{\max,\mathrm{fi}}(G) \mid HM = G\}. \tag{6.1.4}$$

The "co" reminds us of complement. But the property we require is less than being a complement.

Now we state our main tool to prove that $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ is contractible for a finitely generated infinite group G.

Theorem 6.4. (Cone Argument)
Let G be a finitely generated group. The following conditions are equivalent.

(1) For all finite subcomplexes U of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$, there exists a finite index subgroup $K \in \mathscr{H}_{\mathrm{fi}}(G)$ such that $\{K\}$ is not a vertex of U and such that the cone $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$.

(2) For all finite subcomplexes U of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$, there exists a maximal subgroup $M \in \mathscr{H}_{\max,\mathrm{fi}}(G)$ such that $\{M\}$ is not a vertex of U and such that the cone $U * \{M\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$.

(3) The set of subgroups $\mathscr{H}_{coH,\max,\mathrm{fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\mathrm{fi}}(G)$.

If (1), (2), or (3) holds, the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are contractible.

Proof. We start with the proof of the equivalence. Since $\mathscr{H}_{\max,\mathrm{fi}}(G) \subseteq \mathscr{H}_{\mathrm{fi}}(G)$, condition (1) follows from (2).

Suppose that (3) does not hold. Then there exists a subgroup $H \in \mathscr{H}_{\mathrm{fi}}$ such that $\mathscr{H}_{coH,\max,\mathrm{fi}}$ is finite. Let $U = \mathcal{NC}(G, \mathscr{H}')$ with $\mathscr{H}' = \mathscr{H}_{coH,\max,\mathrm{fi}} \cup \{H\}$. Assume that there exists a subgroup $K \in \mathscr{H}_{\mathrm{fi}}$ such that $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. Then $Hg \cap K \neq \varnothing$ for all $g \in G$. Thus $HK = G$. Since K has finite index, there exists a maximal subgroup $M \in \mathscr{H}_{\max,\mathrm{fi}}$, such that $K \leq M$. Hence $HM = G$ and $M \in \mathscr{H}_{coH,\max,\mathrm{fi}}$. Thus $M \in V(U)$. We assumed that $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. Thus $MK = G$, which is not true. Consequently, there is no finite index subgroup K such that $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and (1) does not hold.

Suppose that (3) holds. Let $H_U := \bigcap_{Hg \in V(U)} H$. Since U is finite, H_U is a subgroup of finite index. Therefore there exists a maximal subgroup $M_U \in \mathscr{H}_{coH_U,\max,\mathrm{fi}}$ with $M_U \notin V(U)$. Since $H_U \leq H$ and $G = H_U M_U$, we have $HM_U = G$ for all $Hg \in V(U)$. Lemma 6.3 finishes this direction. Thus (3) follows (2).

If (1) holds, there exists a cone $C(U)$ in $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ for each finite subcomplex U of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. Consequently, every finite subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ is contractible in $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. Therefore $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ is contractible. □

The Cone Argument is a very useful tool to prove that the finite index coset poset is contractible. Maybe more is true and the conditions (1)–(3) are equivalent to $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ being contractible.

It is a difficult task to study the set $\mathscr{H}_{coH,\max,\mathrm{fi}}(G)$ for each $H \in \mathscr{H}_{\mathrm{fi}}(G)$. Thus we consider sets of subgroups, which are easier to study and ensure that $\mathscr{H}_{coH,\max,\mathrm{fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\mathrm{fi}}(G)$.

6.1.1 Normal maximal subgroups

To study $\mathscr{H}_{coH,\max,\text{fi}}(G)$, we search for properties of a maximal subgroup M such that $HM = G$.

Lemma 6.5.
Let M be a maximal subgroup of a finitely generated group G. If M is normal, then $G = HM$ if and only if $H \nleq M$.

Proof. If M is normal, then $M \leq HM \leq G$. Suppose that $H \nleq M$, then $HM \neq M$. Since M is maximal, $G = HM$. If $H \leq M$, then $HM = M$. □

For a finitely generated group G and an arbitrary subgroup H we define the sets

$$\mathscr{H}_{H,\text{fi}}(G) := \{K \in \mathscr{H}_{\text{fi}}(G) \mid H \leq K\} \tag{6.1.5}$$

and

$$\mathscr{H}_{\text{nor},\max}(G) := \{M \in \mathscr{H}_{\max}(G) \mid M \text{ normal subgroup of } G\}. \tag{6.1.6}$$

We call a subgroup $M \in \mathscr{H}_{\text{nor},\max}(G)$ a *normal maximal* subgroup of G. If M is a normal maximal subgroup, then the index $[G : M]$ is finite and a prime number. Thus $\mathscr{H}_{\text{nor},\max}(G) \subseteq \mathscr{H}_{\max,\text{fi}}(G)$. Furthermore, if G is a finitely generated group and H a finite index subgroup, then $\mathscr{H}_{H,\text{fi}}$ is finite.

Theorem 6.6.
Let G be a finitely generated infinite group. If $\mathscr{H}_{\text{nor},\max}(G)$ is infinite, then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ are contractible.

Proof. Let $H \in \mathscr{H}_{\text{fi}}$. If $M \in \mathscr{H}_{\text{nor},\max}$ and $H \nleq M$, then $M \in \mathscr{H}_{coH,\max,\text{fi}}(G)$. Since G is finitely generated, $\mathscr{H}_{H,\text{fi}}$ is finite. Therefore if $\mathscr{H}_{\text{nor},\max}$ is infinite, $\mathscr{H}_{\text{nor},\max} \cap \mathscr{H}_{coH,\max,\text{fi}}$ is infinite. Thus $\mathscr{H}_{coH,\max,\text{fi}}$ is infinite. By the Cone Argument 6.4, we are done. □

Examples of groups with $\mathscr{H}_{\text{nor},\max}(G)$ being infinite are stated in Section 6.2.1.

6.1.2 Coprime index

Lemma 5.28 states that $G = HK$ if and only if $[G : H \cap K] = [G : H][G : K]$. This gives us a second definition of $\mathscr{H}_{coH,\max,\text{fi}}$. Moreover, we use it for the next lemma.

Lemma 6.7.
Let G be a group and H and K subgroups of finite index. If $[G : H]$ and $[G : K]$ are coprime, then $G = HK$.

Proof. Let $a := [G : H]$, $b := [G : K]$, and $c := [G : H \cap K]$. By Lemma 5.27, a divides c. Thus $c = at$. Since b divides c, it divides at. Since a and b are coprime, b divides t. But $c = at \leq ab$. Hence $c = ab$ and $G = HK$. □

For a finitely generated group G and a finite index subgroup H we consider the following subset of $\mathscr{H}_{coH,\max,\text{fi}}(G)$

$$\mathscr{H}_{cop[G:H],\max}(G) := \{M \in \mathscr{H}_{\max,\text{fi}}(G) \mid [G:H],[G:M] \text{ coprime}\}. \qquad (6.1.7)$$

Proposition 6.8.
Let G be a finitely generated group. Suppose that $\mathscr{H}_{cop[G:H],\max}(G)$ is infinite for each $H \in \mathscr{H}_{\text{fi}}(G)$. Then the nerve complex $\mathcal{NC}(G,\mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ are contractible.

Proof. This follows directly from Lemma 6.7 and the Cone Argument 6.4. □

Let $\mathbb{P}$ denote the set of all prime numbers. For a finitely generated group G we define the set

$$\mathscr{H}_{\mathbb{P}}(G) := \{H \in \mathscr{H}_{\text{fi}}(G) \mid [G:H] \in \mathbb{P}\}. \qquad (6.1.8)$$

Each subgroup of prime index in a finitely generated group is a maximal subgroup. Thus $\mathscr{H}_{\mathbb{P}}(G) \subseteq \mathscr{H}_{\max,\text{fi}}(G)$.

For a set of subgroups $\mathscr{H}$ of G we define the set

$$\mathcal{I}(\mathscr{H}) := \{[G:H] \mid H \in \mathscr{H}\}. \qquad (6.1.9)$$

For example, $\mathcal{I}(\mathscr{H}_{\text{nor},\max})$ and $\mathcal{I}(\mathscr{H}_{\mathbb{P}})$ are subsets of $\mathbb{P}$.

Theorem 6.9.
Let G be a finitely generated group. If $\mathscr{H}_{\mathbb{P}}(G)$ is infinite, then the nerve complex $\mathcal{NC}(G,\mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ are contractible.

Proof. Since G is finitely generated, there are only finitely many subgroups with index p. Thus if $\mathscr{H}_{\mathbb{P}}$ is infinite, the set $\mathcal{I}(\mathscr{H}_{\mathbb{P}})$ is infinite. Therefore $\mathscr{H}_{cop[G:H],\max,\text{fi}}$ is infinite for all $H \in \mathscr{H}_{\text{fi}}$. □

For examples of groups with infinitely many prime index subgroups see Section 6.2.2.

For a finitely generated group G we increase the set $\mathscr{H}_{\mathbb{P}}(G)$. Therefore we define $\mathbb{P}^n := \{p^m \mid p \in \mathbb{P}, 1 \leq m \leq n\}$ and the sets of subgroups

$$\mathscr{H}_{\mathbb{P}^n}(G) := \{H \in \mathscr{H}_{\text{fi}}(G) \mid [G:H] \in \mathbb{P}^n\} \qquad (6.1.10)$$

and

$$\mathscr{H}_{\mathbb{P}^n,\max}(G) := \mathscr{H}_{\max,\text{fi}}(G) \cap \mathscr{H}_{\mathbb{P}^n}(G). \qquad (6.1.11)$$

Theorem 6.10.
Let G be a finitely generated group. If $\mathscr{H}_{\mathbb{P}^n,\max}(G)$ is infinite, then the nerve complex $\mathcal{NC}(G,\mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ are contractible.

Proof. Let $\mathscr{H}_{\mathbb{P}^n,\max}$ be infinite. Since G is a finitely generated group, $\mathcal{I}(\mathscr{H}_{\mathbb{P}^n,\max})$ is infinite. Moreover, the power of each prime is bounded. Thus there exist infinitely many prime powers of pairwise different primes. Therefore $\mathscr{H}_{cop[G:H],\max,\text{fi}}$ is infinite for all $H \in \mathscr{H}_{\text{fi}}$. □

We use subgroup graphs to detect the finite index subgroups $M \in \mathscr{H}_{\mathbb{P}^n,\max}(G)$. Therefore it would be easier if it suffices to consider $\mathscr{H}_{\mathbb{P}^n}(G)$.

Lemma 6.11.
Let G be a finitely generated group. The set of subgroups $\mathscr{H}_{\mathbb{P}^n}(G)$ is infinite if and only if $\mathscr{H}_{\mathbb{P}^n,\max}(G)$ is infinite.

Proof. On the one hand $\mathscr{H}_{\mathbb{P}^n,\max}(G) \subseteq \mathscr{H}_{\mathbb{P}^n}(G)$. On the other hand, if a subgroup $H \in \mathscr{H}_{\mathbb{P}^n}$ has index p^m and H is not maximal, then there exists a maximal subgroup M with $H < M$. Thus M has index p^k with $k < m$ and $M \in \mathscr{H}_{\mathbb{P}^n,\max}$. Since $\mathcal{I}(\mathscr{H}_{\mathbb{P}^n})$ consists of prime powers of infinitely many different primes, $\mathscr{H}_{\mathbb{P}^n,\max}$ is infinite if $\mathscr{H}_{\mathbb{P}^n}$ is infinite. □

Corollary 6.12.
Let G be a finitely generated group. If $\mathscr{H}_{\mathbb{P}^n}(G)$ is infinite, then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ are contractible.

For examples of groups with $\mathscr{H}_{\mathbb{P}^n}$ infinite see Section 6.2.3. Thus on the one hand a set of maximal subgroups is again important for the contractibility. On the other hand it suffices to find infinitely many special finite index subgroups. Moreover,

$$\mathscr{H}_{\text{nor},\max}(G) \subseteq \mathscr{H}_{\mathbb{P}}(G) \subseteq \mathscr{H}_{\mathbb{P}^n,\max}(G) \subseteq \mathscr{H}_{\max,\text{fi}}(G).$$

Hence we have created a chain of sets of maximal subgroups. Consequently, $\mathscr{H}_{\max,\text{fi}}(G)$ is infinite for all groups for which we proved that the finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is contractible. This leads to the idea that $\mathscr{C}_{\text{fi}}(G)$ is contractible if $\mathscr{H}_{\max,\text{fi}}(G)$ is infinite. We consider this in Section 7.2.

One can use the idea of the Cone Argument for subsets of $\mathscr{H}_{\text{fi}}$. Therefore we prove the General Cone Argument 8.16 in Section 8.4. It yields the following corollaries.

Corollary 6.13.
Let G be a finitely generated group. If $\mathscr{H}_{\text{nor},\max}(G)$ is infinite, then $\mathcal{NC}(G, \mathscr{H}_{\text{nor},\max})$ is contractible.

Corollary 6.14.
Let G be a finitely generated group. If $\mathscr{H}_{\mathbb{P}}(G)$ is infinite, then $\mathcal{NC}(G, \mathscr{H}_{\mathbb{P}})$ is contractible.

Corollary 6.15.
Let G be a finitely generated group. If $\mathscr{H}_{\mathbb{P}^n}(G)$ is infinite, then $\mathcal{NC}(G, \mathscr{H}_{\mathbb{P}^n})$ and $\mathcal{NC}(G, \mathscr{H}_{\mathbb{P}^n,\max})$ are contractible.

6.2 Examples

In this section we prove that the finite index coset poset is contractible for many infinite finitely generated groups. To prove this, we use the theory of the previous Section 6.1 that if $\mathscr{H}_{\text{nor},\max}(G)$, $\mathscr{H}_{\mathbb{P}}(G)$, or $\mathscr{H}_{\mathbb{P}^n}(G)$ are infinite, then the finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is contractible. In Section 6.2.1 we give examples of finitely generated infinite groups G with $\mathscr{H}_{\text{nor},\max}(G)$ being infinite. These contain free groups, free abelian groups, HNN extensions, $\mathrm{SL}(n, \mathbb{Z})$ with $n \geq 3$, Thompson's group F, Baumslag-Solitar groups, Artin groups, pure braid groups, and infinite virtually cyclic groups of the form $F \rtimes \mathbb{Z}$ with F being a finite group. In Section 6.2.2 we state examples of infinite finitely generated groups G where $\mathscr{H}_{\mathbb{P}}(G)$ is infinite. These contain infinite right angled Coxeter groups, many hyperbolic triangle groups, Fuchsian groups of genus $g \geq 2$, and infinite virtually cyclic groups of the form $A *_C B$. In Section 6.2.3 we state examples of groups with $\mathscr{H}_{\mathbb{P}^n}$ and therefore $\mathscr{H}_{\mathbb{P}^n,\max}$ being infinite. The main examples we give here are the euclidean triangle groups. We prove the existence of the infinitely many subgroups using our theory of subgroup graphs of finite index subgroups.

6.2.1 Infinitely many normal maximal subgroups

We prove that $\mathscr{H}_{\max,\mathrm{fi}}(G)$ is infinite for a finitely generated group $G = \langle X \mid R \rangle$ in the following way. First, we describe connected X-regular graphs Γ_p with p vertices and $(\Gamma_p, v) \cong (\Gamma_p, u)$ for all $u, v \in V(\Gamma_p)$. Then we prove that the graph Γ_p fulfills the defining relators R for infinitely many $p \in \mathbb{P}$. To describe the graphs precisely, we need the following definitions.

Definition 6.16.
Let Γ be an X-graph. The graph $\Gamma|_Y$ with $Y \subseteq X$ is the subgraph of Γ with the following properties: $e \in E(\Gamma|_Y)$ if and only if $e \in E(\Gamma)$ and $\mu(e) = Y$, and $v \in V(\Gamma|_Y)$ if and only if there exists an edge $e \in E(\Gamma|_Y)$ such that $o(e) = v$ or $t(e) = v$.

Suppose that Γ is X-regular, then $V(\Gamma) = V(\Gamma|_Y)$ and $\Gamma|_Y$ is Y-regular. Furthermore, the connected components of $\Gamma|_{\{x\}}$ are (x, n_i)-circles with $|V(\Gamma)| = \sum n_i$.

Definition 6.17. ((a, n)-Circle)
We call the X-regular graph with n vertices and n edges, labeled a, for $a \in X$ the *a-circle of length n* or the *(a, n)-circle*. The graph is shown in Figure 13.

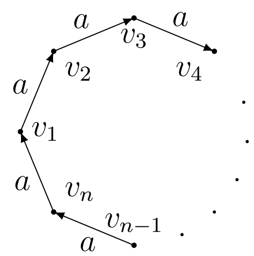

Figure 13: The (a, n)-circle is a connected $\{a\}$-regular graph.

Example 6.18. (Groups with $\mathscr{H}_{\mathbf{nor,max}}$ being infinite)

- *The free group $F(X)$ with finite X.* Let a be a fixed element of X. Let Γ_p be an (a, p)-circle with a loop, labeled x, for each $x \in X \setminus \{a\}$ at each vertex of Γ_p. The graph is shown in Figure 14. The graph Γ_p is X-regular and connected for each $p \in \mathbb{P}$. Thus (Γ_p, v_1) is the subgroup graph of a subgroup H_p of index p in $F(X)$ for each $p \in \mathbb{P}$. Consequently, H_p is a maximal subgroup in $F(X)$. Since (Γ_p, v_1) is isomorphic to (Γ_p, v_i) for all $i = 1, ..., p$ the maximal subgroup H_p is normal. Therefore $\mathscr{H}_{\text{nor,max}}(F(X))$ is infinite.

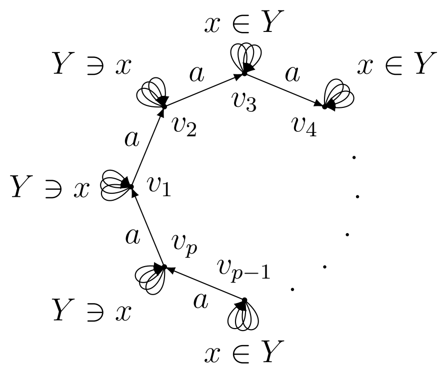

Figure 14: The graph Γ_p for the free group $F(X)$ with a loop, labeled x, for each $x \in Y = X \setminus \{a\}$.

- *The groups $G * \langle a \rangle$, $G \times \langle a \rangle$ and $G \rtimes \langle a \rangle$ generated by $X = Y \sqcup \{a\}$, where $G = \langle\, Y \mid R \,\rangle$ with finite Y.* We use the graph Γ_p of Figure 14.

  Firstly, consider $G * \mathbb{Z} = \langle\, X \mid R \,\rangle$. All relators are words in $Y^{\pm 1}$ and each edge labeled $x \in Y$ is a loop in Γ_p. Consequently, the graph Γ_p fulfills the relators R for each prime $p \in \mathbb{P}$.

  Secondly, $G \times \mathbb{Z} = \langle\, X \mid R' \,\rangle$ with $R' = R \cup \{axa^{-1}x^{-1} \mid x \in Y\}$. Similarly to $G * \mathbb{Z}$, the graph Γ_p fulfills the relators R'. For all $v \in V(\Gamma_p)$ we have $v \xrightarrow{a} v' \xrightarrow{x} v' \xrightarrow{a^{-1}} v \xrightarrow{x^{-1}} v$. Thus Γ_p fulfills the defining relators R' for each $p \in \mathbb{P}$.

  Lastly, $G \rtimes_\psi \mathbb{Z} = \langle\, X \mid R'' \,\rangle$, where $R'' = R \cup \{axa^{-1}(\psi(a)(x))^{-1} \mid x \in Y\}$. Since $\psi(a)(x)$ is a Y-word, we have the path $v \xrightarrow{a} v' \xrightarrow{x} v' \xrightarrow{a^{-1}} v \xrightarrow{(\psi(a)(x))^{-1}} v$ for each $v \in V(\Gamma_p)$. Thus each graph Γ_p fulfills the relators R''.

  Therefore $\mathscr{H}_{\text{nor,max}}(G*\mathbb{Z})$, $\mathscr{H}_{\text{nor,max}}(G\times\mathbb{Z})$, and $\mathscr{H}_{\text{nor,max}}(G\rtimes\mathbb{Z})$ are infinite.

- *Free abelian groups $\mathbb{Z}^n$*, since $\mathbb{Z}^n = \mathbb{Z}^{n-1} \times \mathbb{Z}$.

- *Infinite virtually cyclic groups of the form $F \rtimes \mathbb{Z}$* with F being a finite group.

- *The fundamental group of the Klein bottle*, since it is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}$.

- *HNN extension $G_\alpha = \langle\, Y, a \mid R, aha^{-1}(\alpha(h))^{-1}, \forall h \in H \,\rangle$ with $G = \langle\, Y \mid R \,\rangle$, proper subgroups $H, K < G$ and an isomorphism $\alpha\colon H \to K$.* Let Γ_p be isomorphic to the graph of Figure 14. Since h and $\alpha(h)$ are Y-words and their subpaths are only loops, $v \xrightarrow{a} v' \xrightarrow{h} v' \xrightarrow{a^{-1}} v \xrightarrow{(\alpha(h))^{-1}} v$ for all $v \in V(\Gamma_p)$. Thus Γ_p fulfills the relators for each $p \in \mathbb{P}$. Therefore $\mathscr{H}_{\text{nor,max}}(G_\alpha)$ is infinite.

- *The special linear group* $\mathrm{SL}(n,\mathbb{Z})$ with $n \geq 3$. Let e_{ij} be the elementary matrix with entry 1 in the (i,j)-th place, $i \neq j$. Milnor states in [Mil71, 10.3] that for $n \geq 3$ the group $\mathrm{SL}(n,\mathbb{Z})$ has a presentation with $n(n-1)$ generators e_{ij} and relations

$$\begin{array}{ll} [e_{ij}, e_{kl}] = 1 & \text{if } j \neq k, i \neq l, \\ [e_{ij}, e_{jk}] = e_{ik} & \text{if } i,j,k \text{ are distinct,} \\ (e_{12}e_{21}^{-1}e_{12})^4 = 1. & \end{array}$$

  Let Γ_p be isomorphic to the graph of Figure 14 with labels $a = e_{n-1n}$ and $Y = \{e_{ij} \mid i \neq j, i \neq n-1, j \neq n\}$. Therefore Γ_p fulfills the relators, if e_{n-1n} is not included. Furthermore, Γ_p fulfills the relators $[e_{ij}, e_{n-1n}]$ and $[e_{n-1n}, e_{kl}]$. It remains to check the second kind of relators $[e_{ij}, e_{jk}]e_{ik}^{-1}$. With e_{n-1n} included there are only $[e_{n-1n}, e_{nk}]e_{n-1k}^{-1}$ and $[e_{in-1}, e_{n-1n}]e_{in}^{-1}$. The commutator part of these relators is a circle. Since $i \neq n-1$ and $k \neq n$ the edges with label e_{in} and e_{n-1k} are loops. Thus Γ_p fulfills the defining relators for each $p \in \mathbb{P}$. Hence $\mathscr{H}_{\mathrm{nor,max}}(\mathrm{SL}(n,\mathbb{Z}))$ is infinite.

- *Orientation-preserving Fuchsian groups of genus one.* These are groups $\langle\, X \mid R \,\rangle$ with generators $X = \{a_1, b_1, x_1, ..., x_d, y_1, ..., y_s, z_1, ..., z_t\}$ and relators $R = \{x_1^{m_1}, ..., x_d^{m_d}, x_1 \cdots x_d y_1 \cdots y_s z_1 \cdots z_t [a_1, b_1]\}$ with $d, s, t \geq 0$ and $m_i \geq 2$, see [LS04]. Let Γ_p be isomorphic to the graph of Figure 14 such that $a = a_1$. Hence Γ_p fulfills the relators $x_i^{m_i}$. For $x_1 \cdots x_d y_1 \cdots y_s z_1 \cdots z_t [a_1, b_1]$ we have $v \xrightarrow{x_1 \cdots x_d y_1 \cdots y_s z_1 \cdots z_t} v \xrightarrow{a_1} v' \xrightarrow{b_1} v' \xrightarrow{a_1^{-1}} v \xrightarrow{b_1^{-1}} v$ for different vertices $v, v' \in V(\Gamma_p)$. Thus Γ_p fulfills the relators R for each $p \in \mathbb{P}$. Hence $\mathscr{H}_{\mathrm{nor,max}}$ is infinite.

- *Baumslag-Solitar groups $BS(m,n) = \langle\, a, b \mid ab^n a^{-1} b^{-m} \,\rangle$ for all integers m, n.* Let Γ_p be isomorphic to the graph of Figure 14 with $Y = \{b\}$. Therefore $v \xrightarrow{a} v' \xrightarrow{b^n} v' \xrightarrow{a^{-1}} v \xrightarrow{b^{-m}} v$ for all $v \in V(\Gamma_p)$. Hence the graph Γ_p fulfills the relator for each $p \in \mathbb{P}$ and $\mathscr{H}_{\mathrm{nor,max}}(BS(m,n))$ is infinite.

- *Thompson's group* $F = \langle\, a, x \mid a^{xx}(a^{xa})^{-1}, a^{xax}(a^{xaa})^{-1} \,\rangle$ with $g^h = h^{-1}gh$. Let Γ_p be isomorphic to the graph of Figure 14 with $Y = \{x\}$. Since each edge labeled x is a loop, the important part of a path are the edges with label a. Thus paths with label $x^{-2}ax^2(a^{-1}x^{-1}axa)^{-1}$ and $(xax)^{-1}axax(a^{-2}x^{-1}axa^2)^{-1}$ have the same origin and terminus. Hence Γ_p fulfills the relators for each $p \in \mathbb{P}$. Therefore $\mathscr{H}_{\mathrm{nor,max}}(F)$ is infinite.

- *Artin groups* $A = \langle\, x_1, ..., x_n \mid R \,\rangle$ *(which include all braid groups)* with $R = \{r_{i,j} \mid 1 \leq i < j \leq n\}$ such that $r_{i,j} = \langle x_i, x_j \rangle^{m_{i,j}} (\langle x_j, x_i \rangle^{m_{j,i}})^{-1}$, $m_{i,j} = m_{j,i}$ for $i \neq j \in \{1, ..., n\}$ and $\langle x_i, x_j \rangle^s$ being an alternating product of x_i and x_j of length s starting with x_i. Let Γ_p be an X-regular graph with p vertices $v_1, ..., v_p$ such that the graph $\Gamma_p|_{\{x\}}$ is an (x,p)-circle for each $x \in X$. All (x,p)-circles have the same direction. That is, for all $x \in X$ there is an edge labeled x from v_1 to v_2. The graph Γ_p is shown in Figure 15. Starting in one vertex the reduced path p_r with $\mu(p_r) = r_{i,j}$ leads

$m_{i,j}$ vertices forwards and then $m_{i,j}$ vertices backwards. Hence Γ_p fulfills the relators R for each $p \in \mathbb{P}$. This gives us a set of infinitely many maximal subgroups. Since (Γ_p, v_1) is isomorphic to (Γ_p, v_i) for all $i = 1, ..., p$, the subgroup graphs provide normal subgroups. Thus $\mathscr{H}_{\text{nor,max}}(A)$ is infinite.

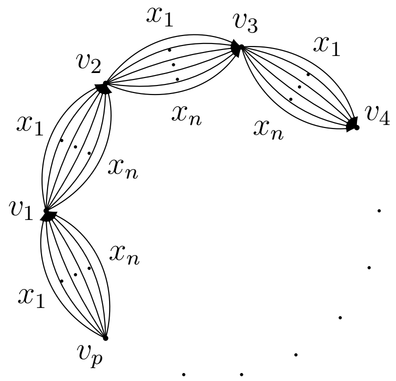

Figure 15: The subgroup graph Γ_p for Artin groups and pure braid groups with generators $X = \{x_1, ..., x_n\}$.

- *Pure braid groups* $PB_m = \langle\, A_{ij}, 1 \leq i < j \leq m \,|\, R_1, R_2, R_3, R_4 \,\rangle$ with relators

  $$\begin{aligned} R_1 &= \{A_{rs}A_{ij}A_{rs}^{-1}A_{ij}^{-1} \mid s < i \text{ or } j < r\}, \\ R_2 &= \{A_{js}A_{ij}A_{js}^{-1}A_{is}^{-1}A_{ij}^{-1}A_{is} \mid i < j < s\}, \\ R_3 &= \{A_{rj}A_{ij}A_{rj}^{-1}A_{ij}^{-1}A_{ir}^{-1}A_{ij}^{-1}A_{ir}A_{ij} \mid i < r < j\}, \\ R_4 &= \{A_{rs}A_{ij}A_{rs}^{-1}A_{is}^{-1}A_{ir}^{-1}A_{is}A_{ir}A_{ij}^{-1}A_{ir}^{-1}A_{is}^{-1}A_{ir}A_{is} \mid i < r < j < s\}, \end{aligned}$$

  see [MKS04]. Let Γ_p be isomorphic to the graph of Figure 15, where n is the number of generators A_{ij} and $X = \{A_{ij} \mid 1 \leq i < j \leq m\}$. If we sum up the exponents of the A_{ij} in each relator $r \in R_i$, it gives 0. Therefore the graph Γ_p fulfills the relators for each $p \in \mathbb{P}$ and $\mathscr{H}_{\text{nor,max}}(PB_m)$ is infinite.

- *The products* $G_1 * ... * G_n$*,* $G_1 \times ... \times G_n$*, and* $G_n \rtimes (G_{n-1} \rtimes (... \rtimes (G_2 \rtimes G_1)...))$ *of finitely generated groups* $G_1, ..., G_n$ *such that* $\mathscr{H}_{\text{nor,max}}(G_1)$ *is infinite.* For the proof see Corollary 6.27.

6.2.2 Infinitely many prime index subgroups

As in the previous section we construct graphs Γ_p and prove that they are subgroup graphs of subgroups of index p for infinitely many primes p. To clarify the structure, we need the following definition.

Definition 6.19. ((a, k, b, l)-Graph)
Let $k, l \geq 2$. We construct the following graph, which we call an (a, k, b, l)*-graph*. We glue an (a, k)-circle with a (b, l)-circle over a single vertex v. We say that the circles share the vertex v. Then we glue this (b, l)-circle with another (a, k)-circle over a different vertex. We glue the second (a, k)-circle with another (b, l)-circle. Repeating these steps we end with a (b, l)-circle. An (a, k)-circle and a (b, l)-circle

share at most one vertex. We add loops with label a or b in such a way that the constructed graph is $\{a, b\}$-regular. Thus, to every vertex of an (a, k)-circle which is not shared with a (b, l)-circle we add a loop labeled b and to every not shared vertex of a (b, l)-circle we add a loop labeled a, see Figure 16.

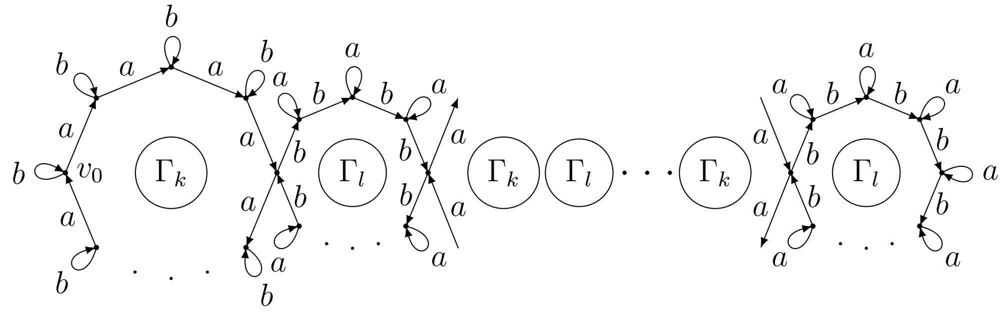

Figure 16: An (a, k, b, l)-graph.

Remark 6.20. Dirichlet's Theorem states that if $x, y \in \mathbb{Z}$ and x and y are coprime, then there exist infinitely many primes of the form $x + ny$ with $n \in \mathbb{N}$. An (a, k, b, l)-graph has $m = k + l - 1 + (k + l - 2)n$ vertices with $n \in \mathbb{N}$. The numbers $k + l - 1$ and $k + l - 2$ are coprime for all $k, l \in \mathbb{N}_{>1}$. Thus by Dirichlet's Theorem, there exist infinitely many $n \in \mathbb{N}$ such that m is prime.

Example 6.21. (Groups with $\mathscr{H}_{\mathbb{P}}$ being infinite)

- All finitely generated infinite groups with $\mathscr{H}_{\text{nor,max}}$ being infinite.
- *The special linear group* $\mathrm{SL}(2, \mathbb{Z}) = \langle\, x, y \mid x^4, y^3, yx^2y^{-1}x^{-2} \,\rangle$. We can take $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Thus $xy = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and xy has infinite order. We consider an $(x, 2, y, 3)$-graph Γ_p with p vertices. See Figure 17 for $p = 7$. The graph Γ_p fulfills the relator $yx^2y^{-1}x^{-2}$, since each path with label x^2 is either an $(x, 2)$-circle or two times a loop. By Remark 6.20, there exist infinitely many primes $p = 4 + 3n$. Hence $\mathscr{H}_{\mathbb{P}}(\mathrm{SL}(2, \mathbb{Z}))$ is infinite.

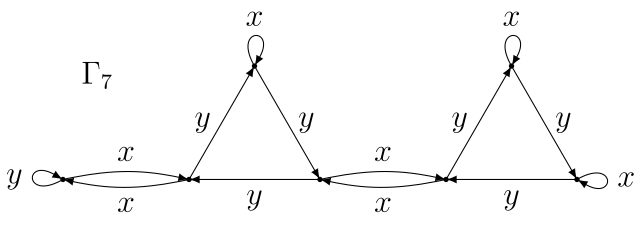

Figure 17: Subgroup graph Γ_7 of $\mathrm{SL}(2, \mathbb{Z})$.

- *The free product* $\mathbb{Z}_s * \mathbb{Z}_t = \langle\, a, b \mid a^s, b^t \,\rangle$ *for* $s, t \in \mathbb{N}_{>1}$. Suppose for $k, l \in \mathbb{N}_{>1}$ that $k|s$ and $l|t$ and that Γ_p is an (a, k, b, l)-graph with p vertices as in Figure 16. The graph Γ_p consists of loops labeled a or b, (a, k)-circles and

(b, l)-circles. Consequently, the graph Γ_p fulfills the relators. By Remark 6.20, there exist infinitely many primes $p = k+l-1+(k+l-2)n$. Thus $\mathscr{H}_{\mathbb{P}}(\mathbb{Z}_s*\mathbb{Z}_t)$ is infinite.

- *The modular group* $\mathrm{PSL}(2,\mathbb{Z})$, since it is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$.

- *The groups* $G*(\mathbb{Z}_s*\mathbb{Z}_t)$, $G\times(\mathbb{Z}_s*\mathbb{Z}_t)$ *and* $G\rtimes_\psi(\mathbb{Z}_s*\mathbb{Z}_t)$ *with* $X = Y \sqcup \{a, b\}$, Y *finite,* $G = \langle\, Y \mid R \,\rangle$, *and* $s, t \in \mathbb{N}_{>1}$. Suppose for $k, l \in \mathbb{N}_{>1}$ that $k|s$ and $l|t$ and that Γ_p is an (a, k, b, l)-graph with p vertices and a loop, labeled x, for each $x \in Y$ at every vertex of Γ_p. As in the example $\mathbb{Z}_s * \mathbb{Z}_t$ the relators a^s and b^t are fulfilled.

  First, consider $G * (\mathbb{Z}_s * \mathbb{Z}_t) = \langle\, X \mid R' \,\rangle$ with $R' = R \cup \{a^s, b^t\}$. All edges labeled $x \in Y$ are loops. Consequently, each Γ_p fulfills the relators R'.

  $G \times (\mathbb{Z}_s * Z_t) = \langle\, X \mid R'' \,\rangle$ with $R'' = R' \cup \{axa^{-1}x^{-1}, bxb^{-1}x^{-1} \mid x \in Y\}$. By the same arguments as for $G * (\mathbb{Z}_s * \mathbb{Z}_t)$ and $G \times \mathbb{Z}$, each graph Γ_p fulfills the relators R''.

  The group $G \rtimes_\psi (\mathbb{Z}_s * \mathbb{Z}_t)$ has $\langle\, X \mid R''' \,\rangle$ as a presentation with relators $R''' = R' \cup \{axa^{-1}(\psi(a)(x))^{-1}, bxb^{-1}(\psi(b)(x))^{-1} \mid x \in Y\}$. By the same arguments as for $G * (\mathbb{Z}_s * \mathbb{Z}_t)$ and $G \rtimes \mathbb{Z}$, each Γ_p fulfills the relators R'''.

  By Remark 6.20, there exist infinitely many primes $p = k+l-1+(k+l-2)n$. Thus $\mathscr{H}_{\mathbb{P}}$ is infinite for the numbered groups.

- *The semidirect product* $\mathbb{Z} \rtimes_\psi G = \langle\, a, Y \mid R \,\rangle$, *where* G *is a finitely generated group.* For the proof see Appendix B.4.

- *Infinite right angled Coxeter groups* $W = \langle\, s_1, ..., s_n \mid s_i^2, (s_is_j)^{m_{i,j}} \,\rangle$. Thus $m_{i,j} \in \{2, \infty\}$ and at least one $m_{i,j} = \infty$. If $m_{i,j} = \infty$, there is no relator. For $m_{\alpha,\beta} = \infty$ let $a := s_\alpha$ and $b := s_\beta$. Let Γ_p be an $(a, 2, b, 2)$-graph with a loop, labeled s_i, at every vertex for all s_i with $i \neq \alpha, \beta$, see Figure 18. Consequently, Γ_p fulfills the relators s_i^2 and $(s_is_j)^2$ with $i, j \neq \alpha, \beta$. We have $v \xrightarrow{a} v' \xrightarrow{a} v$ and $v \xrightarrow{a} v' \xrightarrow{s_i} v' \xrightarrow{a} v \xrightarrow{s_i} v$ and analogously $v \xrightarrow{b^2} v$ and $v \xrightarrow{bs_ibs_i} v$ for all $s_i \neq a, b$. Thus Γ_p fulfills the relators for each $p \in \mathbb{P}_{>2}$. Hence $\mathscr{H}_{\mathbb{P}}$ is infinite.

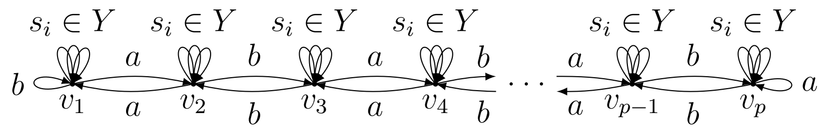

Figure 18: The X-graph Γ_p for an infinite right angled Coxeter group with $Y = X \setminus \{a, b\}$.

- *Infinite Coxeter groups* $W = \langle\, s_1, ..., s_n \mid s_i^2, (s_is_j)^{m_{i,j}} \,\rangle$ *such that* $m_{\alpha,\beta} = \infty$, $m_{\alpha,i}, m_{\beta,i} \in 2\mathbb{N}_{>0} \cup \{\infty\}$, *and arbitrary* $m_{i,j}$ *for* $i, j \neq \alpha, \beta$. We use the same graphs Γ_p as for the right angled Coxeter groups. Therefore Γ_p fulfills s_l^2 and $(s_is_j)^{m_{i,j}}$ for $l \in \{1, ..., n\}$ and $i, j \neq \alpha, \beta$. Since Γ_p fulfills $(s_\alpha s_i)^{m_{\alpha,i}}$

and $(s_\beta s_i)^{m_{\beta,i}}$ for $m_{\alpha,i}, m_{\beta,i} = 2$, it fulfills $m_{\alpha,i}, m_{\beta,i} \in 2\mathbb{N}_{>0}$. Thus $\mathscr{H}_{\mathbb{P}}(W)$ is infinite.

- *The alternating subgroup W^+ of an infinite Coxeter group W as in the previous example.* $W^+ = \langle\, s_k s_i, i \neq k, 1 \leq i \leq n \,|\, (s_k s_i)^{m_{k,i}}, ((s_k s_i)^{-1} s_k s_j)^{m_{i,j}} \,\rangle$ with fixed k such that there are $\alpha, \beta \neq k$ with $m_{\alpha,\beta} = \infty$. Then let $a := s_k s_\alpha$ and $b := s_k s_\beta$ and Γ_p be as in Figure 18 with $Y = \{s_k s_i \mid i \neq \alpha, \beta\}$. Thus Γ_p fulfills the relators $(s_k s_i)^{m_{k,i}}$ and $((s_k s_i)^{-1} s_k s_j)^{m_{i,j}}$ for $i, j \neq \alpha, \beta$. Since $m_{k,\alpha}, m_{k,\beta}, m_{\alpha,i}, m_{\beta,i} \in 2\mathbb{N}_{>0} \cup \{\infty\}$, the graph Γ_p fulfills the relators for each $p \in \mathbb{P}_{>2}$. Thus $\mathscr{H}_{\mathbb{P}}(W^+)$ is infinite.

- *Orientation-preserving Fuchsian groups of genus $g \geq 2$.* These are groups $G = \langle\, X \,|\, R \,\rangle$ with $X = \{a_1, b_1, ..., a_g, b_g, x_1, ..., x_d, y_1, ..., y_s, z_1, ..., z_t\}$ and $R = \{x_1^{m_1}, ..., x_d^{m_d}, x_1 \cdots x_d y_1 \cdots y_s z_1 \cdots z_t [a_1, b_1] \cdots [a_g, b_g]\}$ with $d, s, t \geq 0$, $g \geq 2$, and $m_i \geq 2$, see [LS04]. Let Γ_p be isomorphic to the graph in Figure 18 with $a = a_1$, $b = a_2$ and $Y = X \setminus \{a_1, a_2\}$. Thus Γ_p fulfills the relators $x_i^{m_i}$. Each graph Γ_p fulfills $a_1 a_1^{-1} a_2 a_2^{-1}$ which is the important part of the relator $x_1 \cdots x_d y_1 \cdots y_s z_1 \cdots z_t [a_1, b_1] \cdots [a_g, b_g]$, since all other parts are just loops. Thus Γ_p fulfills the relators R for all odd primes and $\mathscr{H}_{\mathbb{P}}$ is infinite.

- *Non-orientation-preserving Fuchsian groups of genus $g \geq 2$.* These are groups $\langle\, X \,|\, R \,\rangle$ with $X = \{a_1, ..., a_g, x_1, ..., x_d, y_1, ..., y_s, z_1, ..., z_t\}$ and relators $R = \{x_d^{m_d}, ..., x_d^{m_d}, x_1 \cdots x_d y_1 \cdots y_s z_1 \cdots z_t a_1^2 \cdots a_g^2\}$ with $d, s, t \geq 0$, $g \geq 2$, and $m_i \geq 2$, see [LS04]. Let Γ_p be isomorphic to the graph in Figure 18 with $a = a_1$, $b = a_2$ and $Y = X \setminus \{a_1, a_2\}$. Thus the important part of the relator $x_1 \cdots x_d y_1 \cdots y_s z_1 \cdots z_t a_1^2 \cdots a_g^2$ is $v \xrightarrow{a_1} v' \xrightarrow{a_1} v \xrightarrow{a_2} v'' \xrightarrow{a_2} v$. Moreover, Γ_p fulfills all relators $x_i^{m_i}$. Consequently, $\mathscr{H}_{\mathbb{P}}$ is infinite.

- *The hyperbolic triangle groups $\Delta(l, \infty, \infty)$ and $\Delta(l, m, \infty)$ with $l, m \in \mathbb{N}_{>1}$ and related Coxeter groups*, see Proposition C.1 in Appendix C.1.1. Recall that the triangle group $\Delta(l, m, n) = \left\langle\, a, b, c \,|\, a^2, b^2, c^2, (ab)^l, (bc)^m, (ac)^n \,\right\rangle$, where $l, m, n \in \mathbb{N}_{>1} \cup \{\infty\}$ is a Coxeter group.

- *Many hyperbolic triangle groups $\Delta(l, m, n)$ with $l, m, n \in \mathbb{N}_{>1}$*, see Appendix C.1.2.

- *Infinite virtually cyclic groups of the form $A *_{C_A = C_B} B$* with A and B being finite groups and $C_A < A$, $C_B < B$ subgroups of index 2. Since a subgroup of index 2 is normal, we can use Theorem 6.29, which proves that $\mathscr{H}_{\mathbb{P}}(A *_C B)$ is infinite.

- *The amalgamated product $A *_D B$ of two finitely generated groups A and B with $\{1\} \leq D \leq C_A$ and $C_A \cong C_B$ subgroups of index 2 in A and B.* Since a subgroup of index 2 is normal, we can use Theorem 6.29, which proves that $\mathscr{H}_{\mathbb{P}}(A *_D B)$ is infinite.

- *The free product $G_1 * G_2 * ... * G_n$ of finitely generated groups such that $\mathscr{H}_{\text{fi}}(G_i) \neq \varnothing$ for $i = 1, 2$ and $n \geq 2$.* For the proof see Corollary 6.25.

- *The products $G_1 * ... * G_n$, $G_1 \times ... \times G_n$, and $G_n \rtimes (G_{n-1} \rtimes (... \rtimes (G_2 \rtimes G_1)...))$ of finitely generated groups $G_1, ..., G_n$ such that $\mathscr{H}_{\mathbb{P}}(G_1)$ is infinite.* For the proof see Corollary 6.27.

6.2.3 Infinitely many prime power index subgroups

The triangle group $\Delta(l, m, n)$ is a Coxeter group with the following presentation $\langle\, a, b, c \,|\, a^2, b^2, c^2, (ab)^l, (bc)^m, (ac)^n \,\rangle$. It is called euclidean if (l, m, n) equals $(2, 3, 6)$, $(2, 4, 4)$, or $(3, 3, 3)$. For more see Appendix C.

Example 6.22. (Groups with $\mathscr{H}_{\mathbb{P}^n}$ being infinite)

- All finitely generated infinite groups with $\mathscr{H}_{\text{nor,max}}$ or $\mathscr{H}_{\mathbb{P}}$ being infinite.
- *The euclidean triangle groups $\Delta(2, 3, 6)$, $\Delta(2, 4, 4)$, and $\Delta(3, 3, 3)$.* We prove that $\mathscr{H}_{\mathbb{P}^2}$ is infinite for all three of them in Appendix C.2. For $\Delta(2, 3, 6)$ see Figure 40, for $\Delta(2, 4, 4)$ see Figure 39, and for $\Delta(3, 3, 3)$ see Figure 38.
- *The hyperbolic triangle groups $\Delta(2l, 3m, 6n)$, $\Delta(2l, 4m, 4n)$ and $\Delta(3l, 3m, 3n)$ with $l, m, n \in \mathbb{N}_{>0}$.* For the proof that $\mathscr{H}_{\mathbb{P}^2}$ is infinite for all of them, see Appendix C.2.4.
- *Infinite Coxeter groups $W = \langle\, s_1, ..., s_n \,|\, s_i^2, (s_i s_j)^{m_{i,j}}, 1 \leq i < j \leq n \,\rangle$ with $m_{1,k}, m_{2,k}, m_{3,k} \in 2\mathbb{N}_{>0} \cup \{\infty\}$ for all $k = 4, ..., n$ and with $(m_{1,2}, m_{2,3}, m_{1,3})$ equals $(2l, 3m, 6n)$, $(2l, 4m, 4n)$, or $(3l, 3m, 3n)$ for $l, m, n \in \mathbb{N}_{>0}$.* For the proof see Appendix C.2.4.
- *The products $G_1 * ... * G_n$, $G_1 \times ... \times G_n$, and $G_n \rtimes (G_{n-1} \rtimes (... \rtimes (G_2 \rtimes G_1)...))$ of finitely generated groups $G_1, ..., G_n$ such that $\mathscr{H}_{\mathbb{P}^n}(G_1)$ is infinite.* For the proof see Corollary 6.27.

6.3 Inheritance

In the previous section we stated some general results about free, direct, semidirect and amalgamated products. We prove them in this section. Moreover, we increase the list of groups whose finite index coset poset is contractible, by showing the following. The finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ of a finite index subgroup $H < G$ inherits the property of being contractible from $\mathscr{C}_{\text{fi}}(G)$ if $\mathscr{C}_{\text{fi}}(G)$ is contractible by the Cone Argument.

We start with the following result for free products.

Theorem 6.23.
Let G_1 and G_2 be finitely generated groups. Suppose that $\mathscr{H}_{\text{fi}}(G_i) \neq \varnothing$ for $i = 1, 2$. Then the set $\mathscr{H}_{\mathbb{P}}(G_1 * G_2)$ is infinite. Therefore, the nerve complex $\mathcal{NC}(G_1 * G_2, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G_1 * G_2)$ are contractible.

Proof. Let $\langle\, X_i \,|\, R_i \,\rangle$ be a presentation of the group G_i with finite X_i and let $\Gamma(H_i) = \Gamma_{X_i,R_i}(H_i)$ be the subgroup graph of $H_i < G_i$ with index n_i for $i = 1, 2$.

We consider the $(X_1 \cup X_2)$-graph Γ_p with $p = n_1 + n_2 - 1 + (n_1 + n_2 - 2)n$ vertices. The graph Γ_p is built of a copy of the graph $\Gamma(H_1)$ glued with a copy of $\Gamma(H_2)$ at a single vertex v_{11}. This $\Gamma(H_2)$ is glued with a copy of $\Gamma(H_1)$ at another single vertex v_{12} and so on. We end up with a copy of $\Gamma(H_2)$, which is only glued with the previous $\Gamma(H_1)$ over the vertex v_{nn}. At every vertex $v \neq v_{ss}, v_{ss+1}$ of the copies of $\Gamma(H_i)$ we add a loop, labeled x, for each $x \in X_j$ with $i \neq j$. Thus Γ_p is $(X_1 \cup X_2)$-regular. For $\Gamma(H_i)^{(s)}|_{X_i} = \Gamma(H_i)$ the graph Γ_p looks as follows:

$$1_{H_p}\Gamma(H_1)^{(1)}\, v_{11}\, \Gamma(H_2)^{(1)}\, v_{12}\, \Gamma(H_1)^{(2)}\, v_{22} \ldots \Gamma(H_2)^{(n-1)}\, v_{n-1n}\, \Gamma(H_1)^{(n)}\, v_{nn}\, \Gamma(H_2)^{(n)}$$

with $v_{kl} = V(\Gamma(H_i)^{(k)}) \cap V(\Gamma(H_j)^{(l)})$ and $1_{H_p} \in V(\Gamma(H_1)^{(1)})$ is the base-vertex of Γ_p with $v_{11} \neq 1_{H_p}$. Therefore each connected component of $\Gamma_p|_{X_i}$ is either isomorphic to $\Gamma_{X_i,R_i}(H_i)$ or $\Gamma_{X_i,R_i}(G_i)$ for $i = 1, 2$. Hence Γ_p fulfills the relators $r \in R_1 \cup R_2$. Figure 19 shows an example of Γ_p. (Note if $\Gamma(H_i)$ is an (x_i, n_i)-circle, then Γ_p is an (x_1, n_1, x_2, n_2)-graph.)

Since H_1 and H_2 are proper subgroups, $n_1, n_2 > 1$. Thus there exist infinitely many $n \in \mathbb{N}$ such that p is prime. Thus $\mathscr{H}_{\mathbb{P}}(G_1 * G_2)$ is infinite. By Theorem 6.9, we are done. □

Note that G_1 and G_2 can be finite or infinite. Any possible combination works. If G is finite, we know that $\mathscr{H}_{\text{fi}}$ is non-empty if G is not the trivial group. Thus G_i can be any non-trivial finite group. The subgroup H_i, which we choose to construct the graph Γ_p, is allowed to be the trivial subgroup.

This leads to a bunch of examples of groups whose finite index coset poset is contractible.

Example 6.24. Especially, each finitely generated Coxeter group with at least two elements is a simple example for G_1 and G_2. For a finitely generated Coxeter group W with $|W| \geq 2$ we can always choose the alternating subgroup W^+ for the proper finite index subgroup.

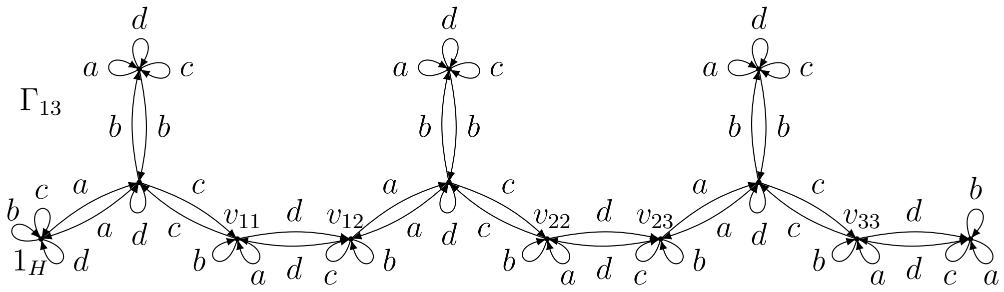

Figure 19: The graph Γ_{13} for $G_1 = \langle\, a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3 \,\rangle$ with $H_1 = \langle\, b, c, abcba \,\rangle$ and $G_2 = \langle\, d \mid d^2 \,\rangle$ with $H_2 = \{1_{G_2}\}$.

To see the construction of Γ_p, we consider the following groups and subgroups. For G_1 we take the euclidean triangle group $\Delta(3, 3, 3)$, which is an infinite Coxeter group and for G_2 the cyclic group $\mathbb{Z}_2$, which is a finite Coxeter group. For the subgroups we choose $H_1 = \langle b, c, abcba\rangle$ a subgroup of index 4 in G_1 and $H_2 = \{1_{G_2}\}$ the trivial subgroup of G_2, which is in this case the alternating subgroup. Then the graph Γ_p has $p = (4+2-1)+(4+2-2)n = 5+4n$ vertices. In Figure 19 the graph Γ_p is shown for $p = 13$.

Corollary 6.25.
Let $G_1, G_2, ..., G_n$ be finitely generated groups with $n \geq 2$ and $\mathscr{H}_{\mathrm{fi}}(G_i) \neq \varnothing$ for $i = 1, 2$. Then the set $\mathscr{H}_{\mathbb{P}}(G_1 * ... * G_n)$ is infinite. Therefore, the nerve complex $\mathcal{NC}(G_1 * ... * G_n, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G_1 * ... * G_n)$ are contractible.

Proof. If $G_i = \langle X_i \mid R_i \rangle$, then $G_1 * ... * G_n = \langle X_1, ..., X_n \mid R_1, ..., R_n \rangle$. Let Γ_p be as in the proof of Theorem 6.23. Thus it suffices to add a loop, labeled x, to every vertex of Γ_p for each $x \in X_3 \cup ... \cup X_n$. □

Therefore the finite index coset poset of a finite free product of at least two groups of Section 6.2 is contractible.

Theorem 6.26.
Let G_1 and G_2 be finitely generated groups. Let G be either $G_1 * G_2$, $G_1 \times G_2$, or $G_2 \rtimes G_1$. Then the following hold.

(1) If $\mathscr{H}_{\mathrm{nor,max}}(G_1)$ is infinite, then $\mathscr{H}_{\mathrm{nor,max}}(G)$ is infinite.

(2) If $\mathscr{H}_{\mathbb{P}}(G_1)$ is infinite, then $\mathscr{H}_{\mathbb{P}}(G)$ is infinite.

(3) If $\mathscr{H}_{\mathbb{P}^n}(G_1)$ is infinite, then $\mathscr{H}_{\mathbb{P}^n}(G)$ is infinite.

Proof. Let $\langle X_i \mid R_i \rangle$ be a presentation of the group G_i with finite X_i for $i = 1, 2$. Let $\Gamma(M) = \Gamma_{X_1,R_1}(M)$ be the subgroup graph of a subgroup M in $\mathscr{H}_{\mathrm{nor,max}}(G_1)$, $\mathscr{H}_{\mathbb{P}}(G_1)$, or $\mathscr{H}_{\mathbb{P}^n}(G_1)$. We add to the graph $\Gamma(M)$ a loop, labeled y, for each $y \in X_2$ at every vertex of $\Gamma(M)$ and call the $(X_1 \cup X_2)$-graph $\Gamma(M)'$. By construction, it is connected and $(X_1 \cup X_2)$-regular. Now we prove that $\Gamma(M)'$ is a subgroup graph of a subgroup of the groups $G_1 * G_2$, $G_1 \times G_2$, and $G_2 \rtimes G_1$.

First, we consider $G_1 * G_2 = \langle X_1, X_2 \mid R_1, R_2 \rangle$. Since $\Gamma(M)'|_{X_1} = \Gamma(M)$, the graph $\Gamma(M)'$ fulfills the relators R_1. Since each edge labeled $y \in X_2$ is a loop, $\Gamma(M)'$ fulfills the relators R_2. Thus the graph $\Gamma(M)'$ provides a finite index subgroup $M' < G_1 * G_2$.

For the groups $G_1 \times G_2 = \langle X_1, X_2 \mid R_1, R_2, xyx^{-1}y^{-1}, x \in X_1, y \in X_2 \rangle$ and $G_2 \rtimes_\psi G_1 = \langle X_1, X_2 \mid R_1, R_2, xyx^{-1}(\psi(x)(y))^{-1}, x \in X_1, y \in X_2 \rangle$ it remains to show that $\Gamma(M)'$ fulfills the relators $xyx^{-1}y^{-1}$ and $xyx^{-1}(\psi(x)(y))^{-1}$, respectively, for all $x \in X_1$ and $y \in X_2$. The image $\psi(x)(y)$ is an X_2-word. Thus a path in $\Gamma(M)'$ with label $(\psi(x)(y))^{-1}$ consists only of loops. Therefore $\Gamma(M)'$ fulfills the relators and provides a subgroup M' of $G_1 \times G_2$ or $G_2 \rtimes G_1$, respectively.

Since $\Gamma(M)'$ has the same number of vertices as $\Gamma(M)$, the index of M in G_1 is the same as M' in G. If the maximal subgroup M is normal, then $(\Gamma(M), v)$ is isomorphic to $(\Gamma(M), w)$ for each $w \in V(\Gamma(M))$. By the construction of $\Gamma(M)'$, we have $(\Gamma(M)', v) \cong (\Gamma(M)', w)$ for all vertices $v, w \in V(\Gamma(M)')$. Thus M' is normal in G if M is normal in G_1. □

By induction over n we prove the following result.

Corollary 6.27.
Let $G_1, ..., G_n$ be finitely generated groups. Let G be either $G_1 * ... * G_n$, $G_1 \times ... \times G_n$, or $G_n \rtimes (G_{n-1} \rtimes (... \rtimes (G_2 \rtimes G_1)...))$. Then the following hold.

(1) If $\mathscr{H}_{\text{nor,max}}(G_1)$ is infinite, then $\mathscr{H}_{\text{nor,max}}(G)$ is infinite.

(2) If $\mathscr{H}_{\mathbb{P}}(G_1)$ is infinite, then $\mathscr{H}_{\mathbb{P}}(G)$ is infinite.

(3) If $\mathscr{H}_{\mathbb{P}^n}(G_1)$ is infinite, then $\mathscr{H}_{\mathbb{P}^n}(G)$ is infinite.

Thus we covered the examples of free, direct, and semidirect products which we stated in the previous subsection.

Corollary 6.28.
Let $G_1, ..., G_n$ be finitely generated groups. Suppose that $\mathscr{H}_{\text{nor,max}}(G_1)$, $\mathscr{H}_{\mathbb{P}}(G_1)$, or $\mathscr{H}_{\mathbb{P}^n}(G_1)$ is infinite. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ are contractible if G is either $G_1 * ... * G_n$, $G_1 \times ... \times G_n$, or $G_n \rtimes (G_{n-1} \rtimes (... \rtimes (G_2 \rtimes G_1)...))$.

As we see, Corollary 6.25 and Corollary 6.28 both consider free products of finitely generated groups. Let us point out the differences. In Corollary 6.25 each group G_i can be finite, but $\mathscr{H}_{\text{fi}}(G_i) \neq \varnothing$ for $i = 1, 2$. In Corollary 6.28 the group G_1 has to be infinite, since $\mathscr{H}_{\mathbb{P}^n}(G_1)$ must be infinite, but $\mathscr{H}_{\text{fi}}(G_i) = \varnothing$ is allowed for $i = 2, ..., n$.

In Example 6.21 of groups with infinitely many prime index subgroups we listed some amalgamated products. We prove the infinity of the $\mathscr{H}_{\mathbb{P}}$ now.

Theorem 6.29.
Let G_1 and G_2 be finitely generated groups. Suppose there exist proper finite index normal subgroups $H_1 \lhd G_1$ and $H_2 \lhd G_2$ such that $H_1 \cong H_2$. Then $\mathscr{H}_{\mathbb{P}}(G_1 *_D G_2)$ is infinite. Thus the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ are contractible for $G = G_1 *_D G_2$ with $\{1\} \leq D \leq H_1$.

Proof. Let $H_1\backslash G_1 = \{H_1, H_1w_2, ..., H_1w_{n_1}\}$ and $H_2\backslash G_2 = \{H_2, H_2u_2, ..., H_2u_{n_2}\}$. Let $\psi\colon H_1 = \langle c_1, ..., c_k\rangle \to H_2$ be an isomorphism. Then there exist presentations $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_2 = \langle X_2 \mid R_2 \rangle$ with generators $X_1 = \{w_2, ..., w_{n_1}, c_1, ..., c_k\}$ and $X_2 = \{u_2, ..., u_{n_2}, \psi(c_1), ..., \psi(c_k)\}$. Let $\Gamma_{X_i,R_i}(H_i)$ be the subgroup graph of H_i in G_i for $i = 1, 2$. Let v be the terminus of the edge e with origin 1_{H_1} and label c_1 in $\Gamma_{X_1,R_1}(H_1)$ and let H_1w_j be the coset corresponding to v. Then $c_1 \in H_1w_j$. Consequently, $v = 1_{H_1}$. Thus e is a loop. Analogously, all edges with label c_i and origin 1_{H_1} or with label $\psi(c_i)$ and origin 1_{H_2} are loops. Since H_1 and H_2 are normal, $(\Gamma_{X_1,R_1}(H_1), 1_{H_1}) \cong (\Gamma_{X_1,R_1}(H_1), v)$ and $(\Gamma_{X_2,R_2}(H_2), 1_{H_2}) \cong (\Gamma_{X_2,R_2}(H_2), v')$ for all vertices v in $\Gamma_{X_1,R_1}(H_1)$ and v' in $\Gamma_{X_2,R_2}(H_2)$. Thus every edge labeled c_i or $\psi(c_i)$ is a loop at each vertex of $\Gamma_{X_1,R_1}(H_1)$ or $\Gamma_{X_2,R_2}(H_2)$, respectively.

For the group $G_1 *_D G_2 = \langle X_1, X_2 \mid R_1, R_2, d\psi(d)^{-1}, d \in D \rangle$ we construct $(X_1 \cup X_2)$-graphs Γ_p as in Theorem 6.23. Thus each Γ_p is a graph with p vertices and a loop for each c_i and $\psi(c_i)$ at every vertex. Since $D \leq H_1$, each $d \in D$ is a word in $\{c_1^{\pm 1}, ..., c_k^{\pm 1}\}$ and each $\psi(d)$ is a word in $\{\psi(c_1)^{\pm 1}, ..., \psi(c_k)^{\pm 1}\}$. Therefore each Γ_p fulfills the relator $d\psi(d)^{-1}$ for all $d \in D$. Thus $\mathscr{H}_{\mathbb{P}}(G_1 *_D G_2)$ is infinite. □

For an example see $\mathrm{SL}(2,\mathbb{Z})$, which is isomorphic to the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Here $G_1 = \langle\, a \,|\, a^4 \,\rangle \cong \mathbb{Z}_4$, $G_2 = \langle\, b \,|\, b^6 \,\rangle \cong \mathbb{Z}_6$, $H_1 = \langle\, a^2 \,\rangle \cong \mathbb{Z}_2$, and $H_2 = \langle\, b^3 \,\rangle \cong \mathbb{Z}_2$.

Remark 6.30. The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\max,\text{fi}})$ is contractible for the special products we considered in the Theorems 6.23–6.28.

Finally, we are interested whether a finite index subgroup inherits the contractibility.

Theorem 6.31.
Let G be a finitely generated group such that, for each $K \in \mathscr{H}_{\text{fi}}(G)$, the set $\mathscr{H}_{coK,\max,\text{fi}}(G)$ is infinite. Let H be a finite index subgroup of G. Then the nerve complex $\mathcal{NC}(H, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(H)$ are contractible.

Proof. Let U be a finite subcomplex of $\mathcal{NC}(H, \mathscr{H}_{\text{fi}})$. Then U is a finite subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. Let $K_U := \bigcap_{Kg \in V(U)} K$. Since U is finite, K_U is a subgroup of finite index. Therefore there exists an $M_U \in \mathscr{H}_{coK_U,\max,\text{fi}}$ such that the cone $U * \{M_U\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. Consequently, the join $U * \{M_U \cap H\}$ is a subcomplex of $\mathcal{NC}(H, \mathscr{H}_{\text{fi}})$ if $M_U \cap H \neq H$. Suppose that $M_U \cap H = H$, then $H \leq M_U$. Thus $K_U M_U \neq G$, since $K_U < H$. Hence $M_U \notin \mathscr{H}_{coK_U,\max,\text{fi}}(G)$. This contradicts the assumption. □

Therefore we obtain a lot more groups whose finite index coset poset is contractible, namely all finite index subgroups of the products of groups stated in this section and the groups of Section 6.2.

7 Non-contractible finite index coset poset

In this section we prove that there exist finitely generated infinite groups whose finite index coset poset is non-contractible. In particular this is the case if $\mathscr{C}_{\mathrm{fi}}(G)$ is empty.

In Section 7.1 we prove that $\mathscr{C}_{\mathrm{fi}}(G)$ is homotopy equivalent to $\mathscr{C}_{\mathrm{fi}}(G/N)$, for any normal subgroup N of G which is contained in the intersection of all maximal subgroups of finite index, see Theorem 7.3. This leads to Theorem 7.4, which states the following. If N is a finite index subgroup, then $\mathscr{H}_{\mathrm{max,fi}}$ is finite and the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ is homotopy equivalent to the coset poset of a finite group, hence non-contractible, and at most $(|\mathscr{H}_{\mathrm{max,fi}}(G)| - 2)$-connected. Groups with $\mathscr{H}_{\mathrm{max,fi}}$ being finite are for example the infinite finitely generated p-groups, which contain the first Grigorchuk group and the Gupta-Sidki p-groups, see Example 7.10.

Therefore we obtain a homotopy invariant for finitely generated infinite groups. In Section 7.2 we study the properties of this invariant. This leads to new questions and the conjecture that the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ of a finitely generated group is contractible if and only if $\mathscr{H}_{\mathrm{max,fi}}(G)$ is infinite.

7.1 The importance of maximal subgroups

In this section we prove the existence of finitely generated infinite groups whose finite index coset poset is non-contractible and give examples.

Lemma 7.1.
Let K be a simplicial complex and L a subcomplex of K. Let $f\colon K \to L$ be simplicial such that $f|_L = \mathrm{id}_L$ and $\sigma \cup f(\sigma)$ is a simplex in K for all simplices $\sigma \in K$. Then f is a homotopy equivalence.

Proof. We write an arbitrary point x in $|K|$ in barycentric coordinates b_v as follows $x = \sum_{v \in V(K)} b_v$. We define $F(x,t) := \sum_{v \in V(K)} b_v((1-t)v + tf(v))$. Then $F(x,0) = \mathrm{id}_K$ and $F(x,1) = f$. Thus F is a homotopy between f and id_K. □

Lemma 7.2. (Correspondence Theorem)
Let G be a group and N a normal subgroup of G. Then there exists a bijection $f\colon \{H \mid N \leq H \leq G\} \leftrightarrow \{H' \mid H' \leq G/N\}$ with $f(H) = H/N = H'$. Moreover, the bijection preserves the index, that is $[G:H] = [G/N : f(H)]$.

Using the Correspondence Theorem we get a bijection $f\colon \mathscr{H}_{N,\mathrm{fi}}(G) \leftrightarrow \mathscr{H}_{\mathrm{fi}}(G/N)$ with $f(H) = H/N$ for any normal subgroup $N \lhd G$.

For a finitely generated group G we define

$$\Phi_{\mathrm{fi}}(G) := \bigcap_{M \in \mathscr{H}_{\mathrm{max,fi}}(G)} M. \tag{7.1.1}$$

Now we can prove the next theorem. Since G is a finitely generated group, $\mathscr{H}_{\mathrm{max,fi}}(G) = \varnothing$ if and only if $\mathscr{H}_{\mathrm{max,fi}}(G) = \varnothing$. In this case $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are empty.

Theorem 7.3.
Let G be a finitely generated group, N a normal subgroup of G, and $N \subseteq \Phi_{\mathrm{fi}}(G)$. Then the map

$$r\colon \mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}}) \to \mathcal{NC}(G, \mathscr{H}_{N,\mathrm{fi}})$$

given by

$$Hg \mapsto \bigcap_{H \leq \bar{H} \in \mathscr{H}_{N,\mathrm{fi}}(G)} \bar{H}g$$

is a deformation retract. Moreover,

$$\Delta\mathscr{C}_{\mathrm{fi}}(G) \simeq \mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}}) \simeq \mathcal{NC}(G/N, \mathscr{H}_{\mathrm{fi}}) \simeq \Delta\mathscr{C}_{\mathrm{fi}}(G/N).$$

Proof. Since $\mathscr{H}_{N,\mathrm{fi}}(G) \subseteq \mathscr{H}_{\mathrm{fi}}(G)$, the nerve complex $\mathcal{NC}(G, \mathscr{H}_{N,\mathrm{fi}})$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. For each $H \in \mathscr{H}_{\mathrm{fi}}(G)$, there exists a maximal subgroup $M \in \mathscr{H}_{\max,\mathrm{fi}}(G)$ such that $H \leq M$. Since N is contained in every maximal subgroup of finite index, we have $\mathscr{H}_{\max,\mathrm{fi}}(G) \subseteq \mathscr{H}_{N,\mathrm{fi}}(G)$. Thus each vertex has an image. Let $\sigma = \{H_0g_0, ..., H_ng_n\}$ be an n-simplex in $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$. Then $\bigcap \sigma \neq \varnothing$ and $r(\sigma) = \{r(H_0g), ..., r(H_ng)\}$. Since $H_ig_i \subseteq r(H_ig_i)$, $r(\sigma)$ and $\sigma \cup r(\sigma)$ are simplices in $\mathcal{NC}(G, \mathscr{H}_{N,\mathrm{fi}})$. Thus r is a simplicial map. If $H \in \mathscr{H}_{N,\mathrm{fi}}(G)$, then $r(Hg) = Hg$. Hence r is a retraction. By Lemma 7.1, r is a deformation retract. Thus $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}}) \simeq \mathcal{NC}(G, \mathscr{H}_{N,\mathrm{fi}})$.

By the Correspondence Theorem, $\mathcal{NC}(G, \mathscr{H}_{N,\mathrm{fi}}) \cong \mathcal{NC}(G/N, \mathscr{H}_{\mathrm{fi}})$. Thus $\Delta\mathscr{C}_{\mathrm{fi}}(G) \simeq \mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}}) \simeq \mathcal{NC}(G/N, \mathscr{H}_{\mathrm{fi}}) \simeq \Delta\mathscr{C}_{\mathrm{fi}}(G/N)$. □

This leads to the special case that $\Phi_{\mathrm{fi}}(G)$ is a finite index subgroup. For an infinite finitely generated group, this is equivalent to $\mathscr{H}_{\max,\mathrm{fi}}(G)$ being finite.

Theorem 7.4.
Let G be a finitely generated group and let $N := \Phi_{\mathrm{fi}}(G)$. Suppose that $\mathscr{H}_{\max,\mathrm{fi}}(G)$ is finite, then the following hold.

(1) G/N is a finite group and therefore $\mathcal{NC}(G/N, \mathscr{H}_{\mathrm{fi}}) = \mathcal{NC}(G/N, \mathscr{H}_{\mathscr{C}})$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G/N) = \Delta\mathscr{C}(G/N)$ are finite and non-contractible.

(2) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ have the homotopy type of the non-contractible finite complexes $\mathcal{NC}(G/N, \mathscr{H}_{\mathscr{C}})$ and $\Delta\mathscr{C}(G/N)$.

(3) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are homotopy equivalent to the finite nerve complex $\mathcal{NC}(G, \mathscr{H}_{\max,\mathrm{fi}})$.

(4) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are at most $(|\mathscr{H}_{\max,\mathrm{fi}}(G)| - 2)$-connected.

Proof. Since $\mathscr{H}_{\max,\mathrm{fi}}(G)$ is finite, N is a normal subgroup of finite index in G. Thus the quotient G/N is a finite group and $\mathcal{NC}(G/N, \mathscr{H}_{\mathrm{fi}}) = \mathcal{NC}(G/N, \mathscr{H}_{\mathscr{C}})$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G/N) = \Delta\mathscr{C}(G/N)$ are finite simplicial complexes. By Theorem 2.6, the complexes $\mathcal{NC}(G/N, \mathscr{H}_{\mathscr{C}})$ and $\Delta\mathscr{C}(G/N)$ are not contractible. By Theorem 7.3, $\Delta\mathscr{C}_{\mathrm{fi}}(G) \simeq \mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}}) \simeq \mathcal{NC}(G/N, \mathscr{H}_{\mathrm{fi}}) \simeq \Delta\mathscr{C}_{\mathrm{fi}}(G/N)$. Therefore we proved (1) and (2).

Since N has finite index and G is finitely generated, $\mathscr{H}_{N,\mathrm{fi}}(G)$ is finite and thus $\mathcal{NC}(G,\mathscr{H}_{N,\mathrm{fi}})$ is finite. Since $\mathscr{H}_{\max,\mathrm{fi}}(G) \subseteq \mathscr{H}_{N,\mathrm{fi}}(G)$, we can use Lemma 2.13. Thus the finite nerve complexes $\mathcal{NC}(G,\mathscr{H}_{N,\mathrm{fi}})$ and $\mathcal{NC}(G,\mathscr{H}_{\max,\mathrm{fi}})$ are homotopy equivalent. Hence $\mathcal{NC}(G,\mathscr{H}_{\mathrm{fi}}) \simeq \mathcal{NC}(G,\mathscr{H}_{\max,\mathrm{fi}})$. Thus we proved (3).

By Theorem 2.18, the finite nerve complex $\mathcal{NC}(G,\mathscr{H}_{\max,\mathrm{fi}})$ is at most $(n-2)$-connected for $n = |\mathscr{H}_{\max,\mathrm{fi}}(G)|$. □

We expect Theorem 7.4(3) to hold also if $\mathscr{H}_{\max,\mathrm{fi}}(G)$ is infinite. Presently we are not able to prove this.

Now we give examples of finitely generated infinite groups with $\mathscr{H}_{\max,\mathrm{fi}}$ being finite.

Example 7.5. (Groups with $\mathscr{H}_{\max,\mathrm{fi}}$ being finite)

- *Finitely generated infinite simple groups G.* Then $\mathscr{H}_{\mathrm{fi}}(G) = \varnothing$. Thus $\mathcal{NC}(G,\mathscr{H}_{\mathrm{fi}})$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are empty. As a convention, this means these complexes are (-2)-connected.

- *The Tarski monster groups*, since they are finitely presented infinite simple groups. For more on Tarski monster groups see Section 8.3.

- *The Thompson's groups T and V*, since they are finitely presented infinite simple groups.

- *The groups $S \times \mathbb{Z}_p$, where S is a finitely generated infinite simple group and p a prime.* Then S is the only finite index subgroup in $S \times \mathbb{Z}_p$. Consequently, $\mathscr{H}_{\max,\mathrm{fi}}(S \times \mathbb{Z}_p) = \{S\}$ and $\Phi_{\mathrm{fi}}(S \times \mathbb{Z}_p) = S$. Thus the nerve complex $\mathcal{NC}(S \times \mathbb{Z}_p, \mathscr{H}_{\mathrm{fi}})$ is homotopy equivalent to $\mathcal{NC}(\mathbb{Z}_p, \mathscr{H}_{\mathrm{fi}})$, which is a simplicial complex with only p vertices. Thus $\mathcal{NC}(S \times \mathbb{Z}_p, \mathscr{H}_{\mathrm{fi}})$ and $\Delta\mathscr{C}_{\mathrm{fi}}(S \times \mathbb{Z}_p)$ are (-1)-connected.

- *The groups $S \times F = \langle X\rangle \times \langle Y\rangle$ with S being a finitely generated infinite simple group and F a non-trivial finite group.* (This is a general version of the previous example.) Let $\Gamma(H)$ be a subgroup graph of a finite index subgroup $H < S \times F$. Since $\mathscr{H}_{\mathrm{fi}}(S) = \varnothing$, each connected component of $\Gamma(H)|_X$ is $\Gamma(S)$. Moreover, $\Gamma(H)|_Y = \Gamma(H')$ with $H' \in \mathscr{H}_{\mathrm{fi}}(F)$. Thus $\left\langle \phi(X), \phi(H'), \phi(g_i X g_i^{-1}), i = 2, ..., n \right\rangle$ is a generating system of H, with $1, g_2, ..., g_n$ a representation system of the cosets of H' in F. Since X and Y commute, $g_i X g_i^{-1} = X$. Thus $\mathscr{H}_{\mathrm{fi}}(S \times F) = \{S \times H' \mid H' \in \mathscr{H}_{\mathrm{fi}}(F)\}$. Therefore the nerve complex $\mathcal{NC}(S \times F, \mathscr{H}_{\mathrm{fi}})$ is homotopy equivalent to $\mathcal{NC}(F, \mathscr{H}_{\mathrm{fi}})$ and $\mathcal{NC}(F/\Phi(F), \mathscr{H}_{\max,\mathrm{fi}})$. Thus $\mathcal{NC}(S \times F, \mathscr{H}_{\mathrm{fi}})$ and $\Delta\mathscr{C}_{\mathrm{fi}}(S \times F)$ have the homotopy type of the coset poset of the finite groups F and $F/\Phi(F)$ and are at most $(|\mathscr{H}_{\max,\mathrm{fi}}(F)| - 2)$-connected.

- *The groups $S * F$, where S is a finitely generated infinite simple group and F a finite group.* By the arguments of the previous example $S \times F$, the set $\mathscr{H}_{\mathrm{fi}}(S * F)$ is finite. Using our theory of subgroup graphs we prove that the finite index subgroup H is maximal in $S * F$ if and only if H' is maximal in F. Thus the subgroup graph $\Gamma(\Phi_{\mathrm{fi}}(S * F))$ is $\Gamma(\Phi(F))$ with a loop, labeled x, for each $x \in X$ at each vertex. Therefore we have $N\backslash(S * F) = \{\Phi_{\text{fi}}(S * F)g_1, ..., \Phi_{\text{fi}}(S * F)g_n\}$ with $g_i = \phi(\mu(p_i))$. Using the subgroup graph $\Gamma(\Phi_{\text{fi}}(S * F))$ we see that we can choose the g_i to be Y-words. Thus $(S * F)/\Phi_{\text{fi}}(S * F) \cong F/\Phi(F)$. Hence $\mathcal{NC}(S * F, \mathscr{H}_{\text{fi}})$ and $\Delta\mathscr{C}_{\text{fi}}(S * F)$ are homotopy equivalent to $\mathcal{NC}(F/\Phi(F), \mathscr{H}_{\text{max,fi}})$ and at most $(|\mathscr{H}_{\text{max,fi}}(F)| - 2)$-connected.

Proposition 7.6.
For all $n \in \mathbb{N}$, there exists an infinite finitely generated group G_n such that $\Delta\mathscr{C}_{\text{fi}}(G_n)$ is n-connected but not $(n+1)$-connected. For example we may put $G_n = S \times (\mathbb{Z}_p)^{n+2}$, with S being a finitely generated infinite simple group.

Proof. As we proved in the example $\Delta\mathscr{C}_{\text{fi}}(S \times (\mathbb{Z}_p)^{n+2}) \simeq \Delta\mathscr{C}_{\text{fi}}((\mathbb{Z}_p)^{n+2})$. We proved in Proposition 2.20 that $\Delta\mathscr{C}_{\text{fi}}((\mathbb{Z}_p)^{n+2})$ is n-connected but not $(n+1)$-connected. □

Thus we proved that for any $n \in \mathbb{N}$, there exists a finite and an infinite finitely generated group such that the finite index coset poset is n-connected but not $(n+1)$-connected.

The groups $S \times F$ and $S * F$ with S being a finitely generated infinite simple group and F a finite group are not residually finite. A *residually finite group* is a group where the intersection of all finite index subgroups is trivial. The following corollary states why we are interested in residually finite groups. It follows directly from the Correspondence Theorem.

Corollary 7.7.
Let G be a finitely generated group and $N := \bigcap_{H \in \mathscr{H}_{\text{fi}}(G)} H$. Then N is normal in G and G/N a residually finite group. Moreover,

$$\mathcal{NC}(G, \mathscr{H}_{\text{fi}}) \cong \mathcal{NC}(G/N, \mathscr{H}_{\text{fi}}).$$

Thus the finite index coset poset of a finitely generated group is homotopy equivalent to the finite index coset poset of a residually finite group. We find residually finite groups whose set of maximal finite index subgroups is finite in the class of finitely generated infinite p-groups.

Lemma 7.8. (Kai-Uwe Bux)
Let G be an infinite finitely generated p-group. Then $\mathscr{H}_{\text{max,fi}}(G)$ is finite.

Proof. Let $H \in \mathscr{H}_{\text{fi}}$. Since G is finitely generated, $N := \bigcap_{g \in G} gHg^{-1}$ is a normal finite index subgroup of G. Therefore G/N is a finite p-group. Since every maximal subgroup of a finite p-group has index p, H/N is a subgroup of a maximal subgroup M/N of index p in G/N. By the Correspondence Theorem, H is a subgroup of a maximal subgroup M of index p in G. Since this holds for all $H \in \mathscr{H}_{\text{fi}}$, each element of the set $\mathscr{H}_{\text{max,fi}}$ is a subgroup of index p. Since G is finitely generated, there are only finitely many subgroups of index p. Thus $\mathscr{H}_{\text{max,fi}}$ is finite. □

Corollary 7.9.
Let G be an infinite finitely generated p-group. Then $G/\Phi_{\mathrm{fi}}(G)$ is a finite p-group and $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ have the homotopy type of a finite wedge of spheres.

Example 7.10. (Finitely generated infinite p-groups)

- *The first Grigorchuk group* $\mathcal{G}_1$ is a residually finite infinite 2-group. The nerve complex $\mathcal{NC}(\mathcal{G}_1, \mathscr{H}_{\mathrm{fi}})$ is homotopy equivalent to the finite nerve complex $\mathcal{NC}((\mathbb{Z}_2)^3, \mathscr{H}_{\mathscr{C}})$, which is homotopy equivalent to $\Delta\mathscr{C}((\mathbb{Z}_2)^3)$. By Proposition 2.20, $\Delta\mathscr{C}((\mathbb{Z}_2)^3)$ is 1-connected. In fact, it has the homotopy type of a wedge of 21 spheres $\mathbb{S}_2$ of dimension 2.

  To prove this, we use the following presentation of $\mathcal{G}_1$

  $$\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, w_n^4, (w_n w_{n+1})^4 (n \geq 0) \rangle$$

  with $w_0 = ad$, $w_{n+1} = \tilde{\sigma}(w_n)$, where $\tilde{\sigma}(a) = aca$, $\tilde{\sigma}(b) = d$, $\tilde{\sigma}(c) = b$, $\tilde{\sigma}(d) = c$, see [dlH00, Section VIII.56]. By Lemma 7.8, the maximal subgroups of finite index have index 2. Using subgroup graphs we prove that there are 7 maximal subgroups, see Figure 20. Using these subgroup graphs we compute

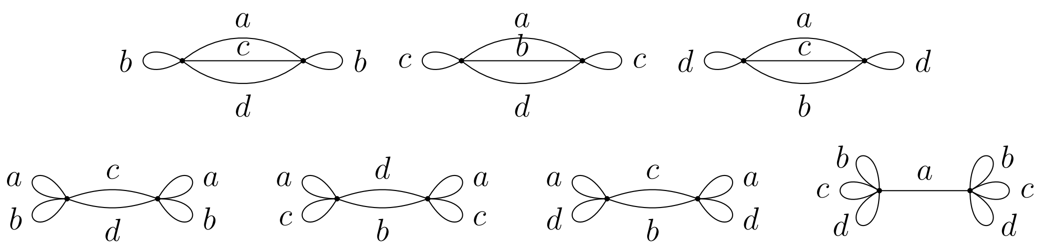

Figure 20: The subgroup graphs of all 7 maximal subgroups of finite index of the first Grigorchuk group. From now on we use undirected edges, labeled x, instead of $(x, 2)$-circles or directed loops, if $x^2 \in R$.

  the intersection $N := \Phi_{\mathrm{fi}}(\mathcal{G}_1)$. The subgroup graph $\Gamma(N)$ has 8 vertices, see Figure 21. Thus $\mathcal{G}_1/N$ is a group of order 8. Studying the subgroup graph $\Gamma(N)$, we obtain $N\backslash\mathcal{G}_1 = \{N, Na, Nc, Nd, Nac, Nad, Ncd, Nacd\}$, which leads to the conclusion that $\mathcal{G}_1/N \cong (\mathbb{Z}_2)^3$. Since $(\mathbb{Z}_2)^3$ is the abelianization of $\mathcal{G}_1$, N is the commutator subgroup. By Proposition 2.20, the coset poset of $(\mathbb{Z}_2)^3$ has the homotopy type of a wedge of 2-spheres. By Lemma 2.16, the number of spheres is $(-1)^2\tilde{\chi}(\mathcal{NC}((\mathbb{Z}_2)^3, \mathscr{H}_{\max}))$, where $\tilde{\chi} = \chi - 1$. We calculate the Euler characteristic $\chi = \sum_{n=0}^{6} (-1)^n k_n$, where k_n denotes the number of n-simplices. Each n-simplex is a set $\{H_0 g, ..., H_n g\}$ with $g \in H := H_0 \cap ... \cap H_n$. Thus for a fixed combination of subgroups $H_0, ..., H_n$ there exist m different n-simplices, with $m := [G : H]$. We have 7 maximal subgroups of index 2, thus $k_0 = 14$. The intersection of two of these maximal subgroups is a subgroup of index 4, thus $k_1 = \binom{7}{2} \cdot 4 = 21 \cdot 4$. For the 2-simplex it is more complicated. There are 7 combinations where the intersection is a subgroup of index 4 and for the remainder the intersection is trivial. Thus we obtain $k_2 = (\binom{7}{3} - 7) \cdot 8 + 7 \cdot 4 = 28 \cdot 8 + 7 \cdot 4$. For the higher dimensional simplices the intersection is always trivial. Thus we obtain $k_n = \binom{7}{n+1} \cdot 8$ for $n = 3, 4, 5, 6$. Therefore we obtain $\chi = 14 - 84 + 252 - 280 + 168 - 56 + 8 = 22$.

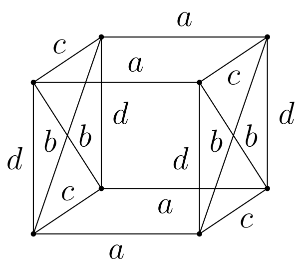

Figure 21: The subgroup graph of $\Phi_{\text{fi}}(\mathcal{G}_1)$.

- *The Gupta-Sidki p-group* $GS_p = \langle a, b \rangle$ *for* $p \in \mathbb{P}_{>2}$ is the subgroup of $\operatorname{Aut}(T)$ generated by the two automorphisms a and b of the p-regular rooted tree T, see [GS83]. The nerve complex $\mathcal{NC}(GS_p, \mathscr{H}_{\text{fi}})$ is homotopy equivalent to the nerve complex $\mathcal{NC}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathscr{H}_{\text{fi}})$. Thus it has the homotopy type of a wedge of 1-spheres and is 0-connected. The number of spheres is $p^3 - p^2 - p + 1$.

  E.L. Pervova proved in [Per05] that GS_p is a group in which all maximal subgroups are normal. There exist only $p + 1$ subgroup graphs with p vertices which are normal and fulfill a^p and b^p. Thus there are at most $p + 1$ maximal subgroups of finite index. The abelianization of GS_p is the group $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence there exists a homomorphism $GS_p \to \mathbb{Z}_p \times \mathbb{Z}_p$. The finite group $\mathbb{Z}_p \times \mathbb{Z}_p$ has $p + 1$ subgroups of index p. The preimage of these groups are subgroups of index p in GS_p. Thus GS_p has $p + 1$ maximal subgroups of index p. They are $\langle a^i b a^{-i} \mid 0 \leq i < p \rangle$, $\langle b^i a b^{-i} \mid 0 \leq i < p \rangle$, and $\langle a^i b a^{-t} a^{-i} \mid 0 \leq i < p \rangle$ for $1 \leq t \leq p - 2$. For example, the subgroup graphs of GS_3 are shown in Figure 22. Since $\Phi(\mathbb{Z}_p \times \mathbb{Z}_p) = \{1\}$ and the

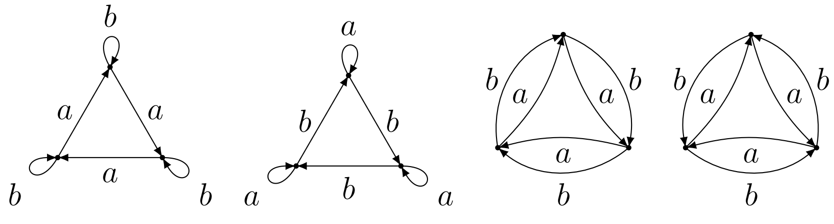

Figure 22: The subgroup graphs of all 4 maximal subgroups of finite index of the Gupta-Sidki 3-group.

preimage of 1 is the commutator subgroup GS_p', we obtain $\Phi_{\text{fi}}(GS_p) = GS_p'$. Now we compute the Euler characteristic $\chi(\mathcal{NC}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathscr{H}_{\max}))$. There are $p + 1$ maximal subgroups in $Z_p \times \mathbb{Z}_p$ and the intersection of at least two of them is always the trivial subgroup. Thus $k_n = \binom{p+1}{n+1} \cdot p^2$ for $0 < n \leq p$ and $k_0 = (p + 1)p$. Thus we have

$$\begin{aligned}
\chi &= \sum_{n=0}^{p}(-1)^n k_n \\
&= p(p+1) + \sum_{n=1}^{p}(-1)^n \binom{p+1}{n+1} p^2 \\
&= p + p^2 - p^2\Big(\sum_{n=1}^{p}(-1)^{n+1}\binom{p+1}{n+1}\Big) \\
&= p + p^2 - p^2\Big(\sum_{n=2}^{p+1}(-1)^{n}\binom{p+1}{n}\Big) \\
&= p + p^2 - p^2\Big(\binom{p+1}{1} - \binom{p+1}{0} + \sum_{n=0}^{p+1}(-1)^{n}\binom{p+1}{n}\Big) \\
&= p + p^2 - p^2\Big(\binom{p+1}{1} - \binom{p+1}{0} + 0\Big) \\
&= p + p^2 - p^2(p+1-1) \\
&= p + p^2 - p^3
\end{aligned}$$

Thus the number of spheres is $(-1)\chi - 1 = 1 - p - p^2 + p^3 = (p-1)^2(p+1)$.

With the examples of Section 6 and this section we obtain the following result for the finite index coset poset.

Corollary 7.11.
There exist examples of finitely generated infinite groups both for contractible and for non-contractible finite index coset posets. Therefore we obtained a non-trivial homotopy invariant.

7.2 A homotopy invariant, a conjecture and questions

In Section 6 and Section 7.1 we proved that there exist examples of finitely generated infinite groups both for contractible and for non-contractible finite index coset posets. Hence we obtained a non-trivial homotopy invariant for finitely generated infinite groups. Thus we are interested in which property of a group provides a (non-)contractible finite index coset poset.

The list of groups whose finite index coset poset is contractible contains hyperbolic and non-hyperbolic, abelian and non-abelian, as well as torsion-free and non-torsion-free groups. The list of groups with non-contractible finite index coset poset includes simple and non-simple, as well as torsion and non-torsion groups. Moreover, the finite index coset poset of $\mathbb{Z}$ is contractible, whereas for the first Grigorchuk group it is not. Both groups are residually finite, Hopfian, amenable, and have a solvable word and conjugacy problem. Moreover, the Thompson's groups $F \subseteq T \subseteq V$ are all of type F_∞ but the finite index coset poset of F is contractible and that of the others is empty.

An obvious difference between the finitely generated groups with contractible and non-contractible finite index coset poset is the cardinality of $\mathscr{H}_{\max,\mathrm{fi}}$. We used this in Theorem 7.4(1), which states if $\mathscr{H}_{\max,\text{fi}}(G)$ is finite, then the finite index coset poset is not contractible. Thus we already have one direction of the following conjecture.

Conjecture 1. *The finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ of a finitely generated infinite group G is contractible if and only if $\mathscr{H}_{\max,\text{fi}}(G)$ is infinite.*

If G is a finitely generated group such that $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\text{nor},\max}$, $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\mathbb{P}}$, or $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\mathbb{P}^n,\max}$, then Conjecture 1 is true.

Recall that we used the Cone Argument 6.4 to prove that a finite index coset poset is contractible. It states that if $\mathscr{H}_{coH,\max,\text{fi}}$ is infinite for all $H \in \mathscr{H}_{\text{fi}}(G)$, then $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ is contractible. Therefore we try to construct a finitely generated infinite group such that $\mathscr{H}_{\max,\text{fi}}$ is infinite but $\mathscr{H}_{coH,\max,\text{fi}}$ is finite for at least one $H \in \mathscr{H}_{\text{fi}}$. If such a group exists, we only know that we cannot use the Cone Argument. Nevertheless, the finite index coset poset can still be contractible. If there does not exist such a group, then we have proved that if $\mathscr{H}_{\max,\text{fi}}$ is infinite, the Cone Argument holds and therefore the finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is contractible.

Question 1. Let $\{I_j \mid j \in J\}$ be a partition of $\mathcal{I}(\mathscr{H}_{\max,\text{fi}}(G))$ with $I_j \subseteq p_j\mathbb{N}$ and $p_j \in \mathbb{P}$. Does there exist a finitely generated infinite group G with $\mathscr{H}_{\max,\text{fi}}$ being infinite such that J can be chosen finite?

If $\mathscr{H}_{\max,\text{fi}}$ is finite, there always exists such a finite partition.

Firstly, we suppose that the answer to Question 1 is negative. Then for every finitely generated group G with $\mathscr{H}_{\max,\text{fi}}(G)$ infinite the following holds. If $\mathcal{I}(\mathscr{H}_{\max,\text{fi}}(G)) = \bigcup_{j\in J} I_j$ with $I_j \subseteq p_j\mathbb{N}$, then J is infinite. Assume now that $\mathscr{H}_{cop[G:H],\max}(G) = \{M_1, ..., M_l\}$, thus is finite, for a subgroup $H < G$ with $[G : H] = n$. Let $p_1, ..., p_k$ be the prime divisors of n. Then we set $I_i := p_i\mathbb{N}$ with $1 \le i \le k$. Thus if $M \in \mathscr{H}_{\max,\text{fi}}(G)$ and $[G : M]$ and $[G : H]$ are not coprime, there exists an i such that p_i divides $[G : M]$ and $[G : M] \in I_i$. Furthermore, we set $I_{k+1} := \{[G : M_1]\}$, ..., $I_{k+l} := \{[G : M_l]\}$. Then we obtain a finite partition as in Question 1, namely $\mathcal{I}(\mathscr{H}_{\max,\text{fi}}) = I_1 \cup ... \cup I_k \cup I_{k+1} \cup ... \cup I_{k+l}$. Thus $\mathscr{H}_{coH,\max,\text{fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\text{fi}}(G)$.

Therefore we have proved that if the answer to Question 1 is negative, then for a group G with $\mathscr{H}_{\max,\text{fi}}(G)$ being infinite it follows that $\mathscr{H}_{cop[G:H],\max}(G)$ is infinite for all $H \in \mathscr{H}_{\text{fi}}(G)$. Thus $\mathscr{H}_{coH,\max,\text{fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\text{fi}}(G)$ and thus $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ is contractible. Therefore, in this case, Conjecture 1 is true.

Secondly, we suppose that the answer to Question 1 is positive. Then $\mathcal{I}(\mathscr{H}_{\max,\text{fi}}) = I_1 \cup ... \cup U_n$. Thus there are at most n different prime indices. Hence $\mathscr{H}_{\mathbb{P}^k,\max}(G)$ is finite and thus $\mathscr{H}_{\text{nor},\max}(G)$ and $\mathscr{H}_{\mathbb{P}}(G)$ are finite. Let $H = M_1 \cap ... \cap M_n$ with $M_i \in \mathscr{H}_{\max,\text{fi}}(G)$ such that $[G : M_i] \in I_i$. Since p_i divides $[G : M_i]$, the index $[G : H]$ is divided by p_i for all $i = 1, ..., n$. Thus $\mathscr{H}_{cop[G:H],\max}(G) = \varnothing$. Hence it is not clear if $\mathscr{H}_{coH,\max,\text{fi}}(G)$ is infinite. Therefore we have to consider $\mathscr{H}_{coH,\max,\text{fi}}(G)$. We have to study maximal finite index subgroups M with $HM = G$, such that M is neither normal nor $[G : M]$ coprime with $[G : H]$. Such subgroups may exist. In this situation Conjecture 1 still may be true.

Conjecture 1 is strong and mostly related to the fact that up to now the finite index coset poset of a finitely generated infinite group was either contractible or homotopy equivalent to the finite index coset poset of a finite group. If one of the following questions can be answered in the affirmative, then Conjecture 1 is not true.

Question 2. Does there exist a finitely generated infinite group G whose finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is neither contractible nor has the homotopy type of the coset poset $\mathscr{C}(F)$ of a finite group F?

This question can be specified to the following.

Question 3. Does there exist a finitely generated infinite group G whose finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is non-contractible and has the homotopy type of an infinite simplicial complex?

Question 4. Does there exist a finitely generated infinite group G whose finite index coset poset $\mathscr{C}_{\text{fi}}(G)$ is non-contractible, has the homotopy type of a finite simplicial complex, but is not homotopy equivalent to the coset poset of a finite group?

If there exists a group G with the properties of Question 2, 3, or 4, then $\mathscr{H}_{\text{max,fi}}(G)$ is infinite.

8 Coset poset

Ramras proved in [Ram05, Sections 3 and 6] that the coset poset of Tarski monster groups and groups whose non-trivial proper subgroups are infinite cyclic is 0-connected but not 1-connected. As far as we know these are the only finitely generated infinite groups whose connectivity of the coset poset is known. In this section we finally study the connectivity and homotopy type of the coset poset of finitely generated infinite groups.

As for the finite index coset poset, we start with the contractible case in Section 8.1. We prove the Exchange Argument, which states the following. If the finite index coset poset is contractible and each subgroup of infinite index is contained in infinitely many finite index subgroups, then the coset poset is contractible. Although $\mathbb{Z}$ is such a group, it leads to only a few examples. Therefore we prove the Intersection Argument, which uses the idea of the Exchange Argument and states the following. If $\mathscr{H}_{H,\text{fi}}$ or $\mathscr{H}_{coH,\max,\text{fi}}$ is infinite and the intersection of the sets $\mathscr{H}_{coH_i,\max,\text{fi}}(G)$ is infinite for all possible sets $\{H_1, ..., H_n\}$ of subgroups H_i which are contained in only finitely many finite index subgroups, then the coset poset is contractible. Examples of such groups are those which have infinitely many normal maximal subgroups.

In Section 8.2 we prove that there exist more finitely generate infinite groups whose coset poset is non-contractible. Using the nerve complex version of Proposition 2.3 we prove that the coset posets of the first Grigorchuk group and the Gupta-Sidki p-groups are non-contractible and homotopy equivalent to the coset posets of finite groups.

In Section 8.3 we study Tarski monster groups, which raises some conjectures. We prove that the coset poset of $G \times \mathbb{Z}_2^n$, with G being a Tarski monster group, has the homotopy type of a wedge of infinitely many $(n+1)$-spheres. Therefore we prove that the coset poset and the finite index coset poset are not always homotopy equivalent. Furthermore, this proves that there exist finitely generated groups G with $\mathscr{H}_{\max}$ being infinite but with non-contractible coset poset $\mathscr{C}(G)$. This leads to the conjecture that the coset poset is non-contractible if and only if $\mathscr{H}_{\max,\text{fi}}$ is finite. Moreover, we expect that the coset poset has the homotopy type of a non-contractible infinite simplicial complex if and only if $\mathscr{H}_{\max,\text{fi}}$ is finite and $\mathscr{H}_{\max}$ infinite. We expect that the coset poset has the homotopy type of the coset poset of a finite group if and only if $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\max}$.

In Section 8.4 we consider higher generation of a set of subgroups $\mathscr{H}$, which is defined using the connectivity of the nerve complex $\mathcal{NC}(G, \mathscr{H})$. With the results of this thesis we list sets of subgroups and their higher generation.

8.1 Contractible coset posets

In this section we prove that there exist finitely generated infinite groups with contractible coset poset. As far as we know this is a new result. The main idea of this section is to exchange vertices of the finite subcomplexes for other vertices such that one can use the Cone Argument.

For G a finitely generated group and an arbitrary subgroup H we define the following set

$$\mathscr{H}_{coH,\max}(G) := \{M \in \mathscr{H}_{\max}(G) \mid HM = G\}. \tag{8.1.1}$$

Certainly there exists a coset poset version of the Cone Argument 6.4. We prove it for the sake of completeness but we will not use it.

Proposition 8.1.
Let G be a finitely generated group. Then the following conditions are equivalent.

(1) For all finite subcomplexes U of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$, there exists a proper subgroup $K < G$ such that $\{K\}$ is not a vertex of U and such that the cone $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$.

(2) For all finite subcomplexes U of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$, there exists a maximal subgroup $M \in \mathscr{H}_{\max}(G)$ such that $\{M\}$ is not a vertex of U and such that the cone $U * \{M\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$.

(3) The set $\mathscr{H}_{coH,\max}(G)$ is infinite for each subgroup $H \in \mathscr{H}_{\mathscr{C}}(G)$.

If (1), (2), or (3) holds, the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are contractible.

Proof. We only prove the equivalence. Since $\mathscr{H}_{\max}(G) \subseteq \mathscr{H}_{\mathscr{C}}(G)$, condition (1) follows from (2).

Suppose that (3) does not hold. Then $\mathscr{H}_{coH,\max}$ is finite for a subgroup $H \in \mathscr{H}_{\mathscr{C}}$. Let $U = \mathcal{NC}(G, \mathscr{H}')$ with $\mathscr{H}' = \mathscr{H}_{coH,\max} \cup \{H\}$. Assume there exists a subgroup $K \in \mathscr{H}_{\mathscr{C}}$ such that $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$. Then $Hg \cap K \neq \varnothing$ for all $g \in G$. Thus $HK = G$. Since G is finitely generated, there exists a maximal subgroup $M \in \mathscr{H}_{\max}$, such that $K \leq M$. Hence $HM = G$ and $M \in \mathscr{H}_{coH,\max}$. Thus $M \in V(U)$. We assumed that $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. Thus $MK = G$, which is not true. Consequently, there is no proper subgroup K such that $U * \{K\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$.

Suppose that (3) holds. Let $H_U := \bigcap_{Hg \in V(U)} H$. Since (3) holds, there exists a maximal subgroup $M_U \in \mathscr{H}_{coH_U,\max}$ with $M_U \notin V(U)$. Since $H_U \leq H$ and $G = H_U M_U$, we have $HM_U = G$ for all $Hg \in V(U)$. Thus (2) follows. □

For several reasons we will not use Proposition 8.1. One reason is that our subgroup graph theory only provides finite index subgroups and we therefore cannot handle $\mathscr{H}_{coH,\max}$. Nevertheless, we find a different way to prove the existence of finitely generated infinite groups whose coset poset is contractible.

To show how our exchange process works, we recall the following definitions of an open star, a star and a link of a simplex σ in a simplicial complex K. The *open star* of a simplex σ in a complex K is

$$St_o(\sigma, K) = \{\tau \in K \mid \sigma \subseteq \tau\}. \tag{8.1.2}$$

The *star* or closed star of σ in K is the set

$$St(\sigma, K) = \{\tau \in K \mid \exists \tau' \in St_o(\sigma, K) : \tau \subseteq \tau'\}. \tag{8.1.3}$$

The *link* of a simplex σ is the set

$$Lk(\sigma, K) = \{\tau \in K \mid \tau \in St(\sigma, K), \tau \cap \sigma = \varnothing\}. \qquad (8.1.4)$$

If $\sigma = \{v\}$, then $St(v, K) = Lk(v, K) \cup St_o(v, K) \simeq Lk(v, K) * \{v\}$.

Proposition 8.2. (Exchange Argument)
Let G be a finitely generated group such that the following hold.

(i) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ is contractible.

(ii) $\mathscr{H}_{K,\text{fi}}(G)$ is infinite for each subgroup $K < G$ of infinite index.

Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are contractible.

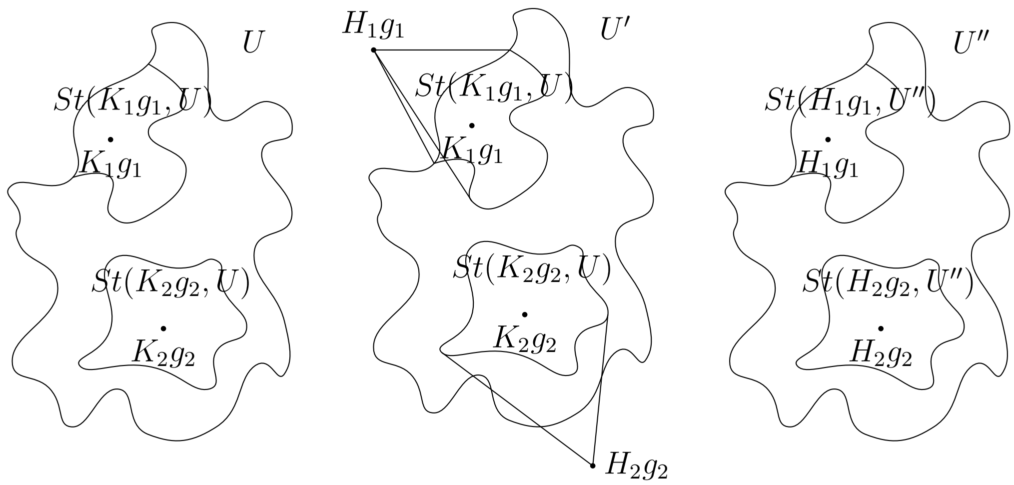

Figure 23: Visualization of the exchange process used in the proof of Proposition 8.2. Recall that $Lk(K_1g_1, U) = Lk(H_1g_1, U'')$, $St(K_1g_1, U) = Lk(H_1g_1, U')$, and $Lk(K_1g_1, U') = St(H_1g_1, U'')$.

Proof. Let U be a finite subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$. If U is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$, we are done.

Suppose that $K_1g_1, ..., K_ng_n$ are all vertices of U such that K_i has infinite index. Note that $K_i = K_j$ is possible and allowed. Then $\mathscr{H}_{K_i,\text{fi}}(G)$ is infinite for $i = 1, ..., n$. Thus there exist n pairwise different subgroups $H_i \in \mathscr{H}_{\text{fi}}(G)$ such that $K_i < H_i$ and $H_ig_i \notin V(U)$ for $i = 1, ..., n$. We now replace K_ig_i by H_ig_i. Let

$$U' := U \cup \bigcup_{i=1}^{n} \{\sigma \cup \{H_ig_i\} \mid \sigma \in St(K_ig_i, U) \cup \varnothing\}.$$

Since $K_i < H_i$, we have $\bigcap \sigma \cap H_ig_i \neq \varnothing$ for each $\sigma \in St(K_ig_i, U)$. Thus the simplicial complex U' is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$. Consider the simplicial map $f\colon U' \to U'$ defined by $f(K_ig_i) = H_ig_i$ and $f(Hg) = Hg$ for all $K_ig_i \neq Hg \in U'$. Let $U'' := \text{im} f$. Then U'' is a deformation retract of U'. Moreover, we replaced K_ig_i by H_ig_i. Since $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ is contractible, the subcomplex U'' is contractible in $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. Since U'' is a deformation retract of U', U' is contractible in $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$. Hence U is contractible in $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$. □

The easiest example is the group $\mathbb{Z}$, since $\mathscr{H}_{\mathscr{C}}(\mathbb{Z}) = \mathscr{H}_{\text{fi}}(\mathbb{Z}) \cup \{1\}$. Therefore we study semidirect products $\mathbb{Z} \rtimes_\psi F$ with F being a finite group in Appendix B.3. There we prove that $\mathbb{Z} \rtimes F$ is a finite group with the properties of the Exchange Argument.

Remark 8.3. We proved that the coset poset $\mathscr{C}(\mathbb{Z} \rtimes_\psi F)$ is contractible for any finite group F. In particular, we proved that the coset posets $\mathscr{C}(\mathbb{Z} \rtimes_\psi \mathbb{Z}_2)$ and $\mathscr{C}(\mathbb{Z})$ are contractible. This extends [Ram05, Remark 3.9], where Ramras states that $\mathscr{C}(\mathbb{Z} \rtimes \mathbb{Z}_2)$ and $\mathscr{C}(\mathbb{Z})$ are simply connected.

Note the following. Since each subgroup of infinite index is a subgroup of a finite index subgroup, $\mathscr{H}_{\max}(\mathbb{Z} \rtimes_\psi F) = \mathscr{H}_{\max,\text{fi}}(\mathbb{Z} \rtimes_\psi F)$. Thus $\mathcal{NC}(\mathbb{Z} \rtimes_\psi F, \mathscr{H}_{\max})$ is contractible.

Proposition 8.4. (Intersection Argument)
Let G be a finitely generated group such that the following hold.

(i) $\mathscr{H}_{H,\text{fi}}(G)$ or $\mathscr{H}_{coH,\max,\text{fi}}(G)$ is infinite for each subgroup $H \in \mathscr{H}_{\mathscr{C}}(G)$.

(ii) For every finite set of subgroups $\{H_1, ..., H_n\} \subseteq \mathscr{H}_{\mathscr{C}}(G)$ the intersection $\bigcap_{i=1}^{n} \mathscr{H}_{coH_i,\max,\text{fi}}$ is infinite if $\mathscr{H}_{H_i,\text{fi}}$ is finite for $i = 1, ..., n$.

Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are contractible.

Proof. Since $\mathscr{H}_{H,\text{fi}}$ is finite for all $H \in \mathscr{H}_{\text{fi}}$, property (i) includes the property that $\mathscr{H}_{coH,\max,\text{fi}}$ is infinite for all $H \in \mathscr{H}_{\text{fi}}$. Let U be a finite subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$. As in the proof of the Exchange Argument we replace each vertex Kg with $\mathscr{H}_{K,\text{fi}}$ being infinite by a vertex Hg with H being a finite index subgroup. We denote the new subcomplex with U'. Let $H_1g_1, ..., H_ng_n$ be the vertices of U'. Then $\mathscr{H}_{H_i,\text{fi}}$ is finite for $i = 1, ..., n$. By (ii), the intersection $\bigcap_{i=1}^{n} \mathscr{H}_{coH_i,\max,\text{fi}}$ is infinite. Thus there exists a maximal finite index subgroup M such that $U' * \{M\}$ is a cone. Hence U' and U are contractible in $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$. □

One might ask why the intersection has to be infinite and why non-empty is not enough. Suppose that $H_1, ..., H_n$ are subgroups such that $\mathscr{H}_{H_i,\text{fi}}$ are finite and $\bigcap_{i=1}^{n} \mathscr{H}_{coH_i,\max,\text{fi}} = \{M_1, ..., M_m\}$. Since M_i is maximal, $\mathscr{H}_{M_i,\text{fi}} = \{M_i\}$. Consider now $H_1, ..., H_n, M_1, ..., M_m$. Then $\bigcap_{i=1}^{n} \mathscr{H}_{coH_i,\max,\text{fi}} \cap \bigcap_{i=1}^{m} \mathscr{H}_{coM_i,\max,\text{fi}} = \varnothing$.

With the following theorem and the examples of Section 6.2.1 we obtain a large class of groups whose coset poset is contractible.

Theorem 8.5.
Let G be a finitely generated group. If $\mathscr{H}_{\text{nor},\max}(G)$ is infinite, then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are contractible.

Proof. We prove that the Intersection Argument holds for such groups. Let $\mathscr{H}_{coH,\text{nor},\max}(G) := \mathscr{H}_{\text{nor},\max}(G) \cap \mathscr{H}_{coH,\max,\text{fi}}(G)$ for a subgroup $H < G$. By

Lemma 6.5, $M \in \mathscr{H}_{coH,\text{nor},\max}$ if and only if $H \not\leq M$. Thus if $\mathscr{H}_{H,\text{fi}}$ is finite and $\mathscr{H}_{\text{nor},\max}$ infinite, then $\mathscr{H}_{coH,\text{nor},\max}$ is infinite.

Let $H_1, ..., H_n$ be subgroups, where $\mathscr{H}_{H_i,\text{fi}}$ is finite. Let $M \in \mathscr{H}_{coH_1,\text{nor},\max}$. Suppose that $M \notin \mathscr{H}_{coH_2,\text{nor},\max}$. Then $H_2 \leq M$. Since $\mathscr{H}_{H_2,\text{fi}}$ is finite, there are only finitely many such M. Thus $\mathscr{H}_{coH_1,\text{nor},\max} \cap \mathscr{H}_{coH_2,\text{nor},\max}$ is infinite. By induction, the intersection $\bigcap_{i=1}^{n} \mathscr{H}_{coH_i,\max,\text{fi}}$ is infinite. □

For examples of groups where $\mathscr{H}_{\text{nor},\max}(G)$ is infinite see Section 6.2.1. They contain Artin groups, pure braid groups, $G * \mathbb{Z}$, $G \times \mathbb{Z}$, and $G \rtimes \mathbb{Z}$ for any finitely generated group G, and thus the free groups, free abelian groups and the fundamental group of the Klein bottle. Note that $\mathbb{Z} \rtimes F$ is not in the list of such groups if F is non-trivial and ψ not the identity, see Proposition B.4.

As in Section 6.3 some products inherit the property of contractibility. The next corollary follows directly from Theorem 6.26.

Corollary 8.6.
Let $G_1, ..., G_n$ be finitely generated groups. Suppose that $\mathscr{H}_{\text{nor},\max}(G_1)$ is infinite. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are contractible if G is $G_1 \times ... \times G_n$, $G_1 * ... * G_n$, or $G_n \rtimes (G_{n-1} \rtimes (... \rtimes (G_2 \rtimes G_1)...))$.

Recall Proposition 2.4 which states that $\mathscr{C}(G) \simeq \mathscr{C}(G/N) * \mathscr{C}(G, N)$. Thus if $\mathscr{C}(G/N)$ is contractible, then $\mathscr{C}(G)$ is contractible. Hence if G is a finitely generated infinite group and there exists a quotient G/N such that $\mathscr{H}_{\text{nor},\max}(G/N)$ is infinite, then $\mathscr{C}(G)$ is contractible.

8.2 Non-contractible coset posets

In this section we consider the case where the coset poset is non-contractible. Ramras already proved that such groups exist. We show that there exist groups whose coset poset has the homotopy type of the coset poset of a finite group.

The following proposition proves that the coset poset of infinite groups is 0-connected.

Proposition 8.7. (See [Bro00, Proposition 14])
$\mathscr{C}(G)$ is connected unless G is cyclic of prime power order.

K.S. Brown proved in [Bro00, Proposition 8] that $\mathscr{C}(G) \simeq \mathscr{C}(G/N)$ if G is a finite group and N is a normal subgroup of G contained in the Frattini subgroup $\Phi(G) = \bigcap_{M \in \mathscr{H}_{\max}(G)} M$. In [Ram05, Section 3] D.A. Ramras states that this can be extended to infinite groups with the same proof. The nerve complex version of it can be either proved using the fact that $\mathscr{C}(G) \simeq \mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ or by a similar proof as for Theorem 7.3.

For a finitely generated group G and an arbitrary subgroup H we define

$$\mathscr{H}_H(G) = \{K \in \mathscr{H}_{\mathscr{C}}(G) \mid H \leq K\}. \quad (8.2.1)$$

Note that if H has finite index, then $\mathscr{H}_H = \mathscr{H}_{H,\text{fi}}$.

Theorem 8.8.
Let G be a finitely generated group, N a normal subgroup of G, and $N \subseteq \Phi(G)$. Then the map

$$r\colon \mathcal{NC}(G, \mathscr{H}_{\mathscr{C}}) \to \mathcal{NC}(G, \mathscr{H}_N) \cong \mathcal{NC}(G/N, \mathscr{H}_{\mathscr{C}})$$

given by

$$Hg \mapsto \bigcap_{H \leq \bar{H} \in \mathscr{H}_N} \bar{H}g$$

is a deformation retract.

If N has finite index, then $\Phi(G)$ is of finite index in G. Since G is a finitely generated group, $\Phi(G)$ has finite index if and only if $\mathscr{H}_{\max} = \mathscr{H}_{\max,\text{fi}}$ and thus $\Phi(G) = \Phi_{\text{fi}}(G)$.

Corollary 8.9.
Let G be a finitely generated infinite group such that $\mathscr{H}_{\max,\text{fi}}(G)$ is finite and $\mathscr{H}_{\max}(G) = \mathscr{H}_{\max,\text{fi}}(G)$. Then the following hold.

(1) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are homotopy equivalent to the finite non-contractible complexes $\mathcal{NC}(G/\Phi(G), \mathscr{H}_{\mathscr{C}})$ and $\Delta\mathscr{C}(G/\Phi(G))$.

(2) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are homotopy equivalent to the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\max})$.

(3) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ and the order complex $\Delta\mathscr{C}(G)$ are at most $(|\mathscr{H}_{\max}(G)| - 2)$-connected.

Note that we have not proved that the coset poset $\mathscr{C}(G)$ is non-contractible if $\mathscr{H}_{\max}$ is finite. But we expect this to be true.

Example 8.10. (Groups with $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\max}$)

- *The first Grigorchuk group $\mathcal{G}_1$* is a group in which each maximal subgroup is normal, see [Per00]. Thus $\Phi(\mathcal{G}_1) = \Phi_{\text{fi}}(\mathcal{G}_1)$. In Example 7.10 we proved that $\Phi_{\text{fi}}(\mathcal{G}_1)$ is the commutator subgroup and thus $\mathcal{G}_1/\Phi_{\text{fi}}(\mathcal{G}_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence the simplicial complexes $\mathcal{NC}(\mathcal{G}_1, \mathscr{H}_{\mathscr{C}})$ and $\Delta\mathscr{C}(\mathcal{G}_1)$ have the homotopy type of a wedge of 21 spheres $\mathbb{S}_2$ of dimension 2. Hence they are 1-connected.

- *The Gupta-Sidki p-group GS_p for $p \in \mathbb{P}_{\geq 2}$* are groups in which each maximal subgroup is normal, see [Per05]. We proved that $GS_p/\Phi(GS_p) \cong (\mathbb{Z}_p)^2$. Thus the simplicial complexes $\mathcal{NC}(GS_p, \mathscr{H}_{\mathscr{C}})$ and $\Delta\mathscr{C}(GS_p)$ have the homotopy types of wedges of $p^3 - p^2 - p + 1$ spheres $\mathbb{S}_1$ of dimension 1. Hence they are 0-connected.

We give more examples of finitely generated infinite groups with non-contractible coset posets in Section 8.3.

8.3 Tarski monster groups, conjectures and questions

In this section we study the importance of the maximal subgroups for the homotopy type of the coset poset. Therefore we study Tarski monster groups. This leads to some conjectures and questions.

We would like to determine a connection between algebraic properties of a finitely generated group and the connectivity of the coset poset. This is possible for the class of groups in which each maximal subgroup is normal, which is studied in [Myr15]. Examples of such groups are abelian or nilpotent groups. Moreover, a linear group G has $\mathscr{H}_{\mathrm{nor,max}} = \mathscr{H}_{\max}$ if and only if it is virtually solvable. By Corollary 8.9 and Theorem 8.5, we obtain the following result.

Corollary 8.11.
Let G be a finitely generated group with $\mathscr{H}_{\mathrm{nor,max}}(G) = \mathscr{H}_{\max}(G)$. Then the coset poset $\mathscr{C}(G)$ and the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ are non-contractible if and only if $\mathscr{H}_{\mathrm{nor,max}}(G)$ is finite.

We know that the coset poset of an infinitely generated group is contractible. Thus if Corollary 8.11 holds for all groups, then $\mathscr{H}_{\mathrm{nor,max}}$ is infinite for an infinitely generated group where each maximal subgroup is normal. Thus we ask the following.

Question 5. Does Corollary 8.11 hold for all groups?

Conjecture 1 assumes that the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ of a finitely generated group G is non-contractible if and only if $\mathscr{H}_{\mathrm{max,fi}}$ is finite. Since $\mathscr{C}(G)$ contains all subgroups one might ask the following. Is the coset poset $\mathscr{C}(G)$ contractible if and only if $\mathscr{H}_{\max}$ is infinite? We can answer this negatively and prove this by studying Tarski monster groups.

Let G be a Tarski monster group. This is an infinite p-group where each non-trivial proper subgroup has order p. Therefore $\mathscr{H}_{\max}(G) = \mathscr{H}_{\mathscr{C}}(G) \setminus \{1_G\}$. The order complex $\Delta\mathscr{C}(G)$ and the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ are 0-connected but not 1-connected. By Proposition 8.7, the coset poset $\mathscr{C}(G)$ is 0-connected. As Ramras notes in [Ram05, Section 3] the coset poset is not 1-connected. To prove this, we consider the path

$$\{1_G\} \subseteq \langle g \rangle \supseteq \{g\} \subseteq \langle h \rangle g \supseteq \{hg\} \subseteq \langle hg \rangle \supseteq \{1_G\},$$

which is a non-contractible circle if $h \notin \langle g \rangle$.

Thus there exist infinite groups with infinitely many maximal subgroups but non-contractible coset poset.

Conjecture 2. *The coset poset $\mathscr{C}(G)$ of a finitely generated group G is non-contractible if and only if $\mathscr{H}_{\mathrm{max,fi}}(G)$ is finite.*

Since Conjectures 1 and 2 are strong, one might ask the following.

Question 6. Is the coset poset $\mathscr{C}(G)$ of a finitely generated group G contractible if and only if the finite index coset poset $\mathscr{C}_{\mathrm{fi}}(G)$ is contractible?

In the previous sections the coset poset $\mathscr{C}(G)$ of a finitely generated infinite group G was either contractible or homotopy equivalent to the coset poset $\mathscr{C}(F)$ of a finite group F. To prove that there are other kinds of coset posets, we study the coset poset of a Tarski monster group closer. Let $H := \langle h \rangle$ be a non-trivial subgroup and $H \backslash G = \{Hg_i \mid i \in I\}$ such that $1 \in I$ and $g_1 = h$. Thus I is infinite. We consider the following path for each $i \in I$:

$$\{1_G\} \subseteq \langle g_i \rangle \supseteq \{g_i\} \subseteq \langle h \rangle g_i \supseteq \{hg_i\} \subseteq \langle hg_i \rangle \supseteq \{1_G\}.$$

Hence we have infinitely many circles. Thus the order complex $\Delta\mathscr{C}(G)$ has the homotopy type of a wedge of infinitely many 1-spheres.

Proposition 8.12.
For each $n \in \mathbb{N}$, there exists a finitely generated infinite group G_n such that the coset poset $\mathscr{C}(G_n)$ is n-connected but not $(n+1)$-connected and has the homotopy type of an infinite simplicial complex. For example take $G_n = G \times (\mathbb{Z}_2)^n$, where G is a Tarski monster group.

Proof. Let $G_n = G \times (\mathbb{Z}_2)^n$, where G is a Tarski monster group. By Proposition 2.4, $\mathscr{C}(G_n) \simeq \mathscr{C}(G_n/N) * \mathscr{C}(G_n, N)$ for a normal subgroup N of G_n. We choose $N = \mathbb{Z}_2 = \langle a \rangle$. Thus $G_n/\mathbb{Z}_2 \cong G_{n-1}$ and $\mathscr{C}(G_n, \mathbb{Z}_2) = \{G_{n-1}, G_{n-1}a\}$. Therefore $\mathscr{C}(G_n) \simeq \mathscr{C}(G_{n-1}) * \{G_{n-1}, G_{n-1}a\} \simeq \mathrm{Susp}(\mathscr{C}(G_{n-1}))$. Furthermore, we proved that $\mathscr{C}(G_0) = \mathscr{C}(G) \simeq \bigvee_\infty \mathbb{S}_1$. Hence we have $\mathscr{C}(G_1) \simeq \mathrm{Susp}(\bigvee_\infty \mathbb{S}_1) \simeq \bigvee_\infty \mathbb{S}_2$. By induction we obtain $\mathscr{C}(G_n) \simeq \bigvee_\infty \mathbb{S}_{n+1}$. □

With these results we expect the following.

Conjecture 3. *The coset poset $\mathscr{C}(G)$ of a finitely generated group G is non-contractible and has the homotopy type of an infinite simplicial complex if and only if $\mathscr{H}_{\max,\mathrm{fi}}(G)$ is finite and $\mathscr{H}_{\max}(G)$ is infinite.*

With Conjecture 2 this leads to the following.

Conjecture 4. *The coset poset $\mathscr{C}(G)$ of a finitely generated group G is non-contractible and has the homotopy type of a finite simplicial complex if and only if $\mathscr{H}_{\max}(G)$ and $\mathscr{H}_{\max,\mathrm{fi}}(G)$ are finite.*

If Conjectures 1–4 are true, there is still the question if the finite simplicial complex can be a coset poset of a finite group. We expect the following.

Conjecture 5. *The coset poset $\mathscr{C}(G)$ of a finitely generated group G has the homotopy type of the coset poset $\mathscr{C}(F)$ of the finite group F, and thus is non-contractible if and only if $\mathscr{H}_{\max}(G) = \mathscr{H}_{\max,\mathrm{fi}}(G)$ and $\mathscr{H}_{\max}$ is finite. By Theorem 8.8, we can choose $G/\Phi(G)$ for F.*

All in all we have the following main results for the contractibility of the coset poset and the finite index coset poset.

Corollary 8.13.

(1) There exist examples of finitely generated infinite groups both for contractible and for non-contractible coset posets and finite index coset posets.

(2) There exist finitely generated infinite groups G such that $\mathscr{C}(G)$ and $\mathscr{C}_{\text{fi}}(G)$ are not homotopy equivalent, e.g. for G being a Tarski monster group.

(3) The coset poset and the finite index coset poset are two different non-trivial homotopy invariants.

(4) There exist examples of finitely generated infinite groups both for coset posets with the homotopy type of a finite complex and for coset posets with the homotopy type of an infinite complex.

8.4 Higher generation

H. Abels and S. Holz introduced the simplicial complex $\mathcal{NC}(G, \mathscr{H})$, which we call the nerve complex, in [AH93] to define higher generation of a family $\mathscr{H}$ of subgroups of a group G. In this section we translate our results about the homotopy type of the nerve complex to higher generation of sets of subgroups. Moreover, we prove more homotopy equivalences of nerve complexes.

Definition 8.14.
Let $\mathscr{H}$ be a family of subgroups of a group G. The family $\mathscr{H}$ is called *n-generating* if $\mathcal{NC}(G, \mathscr{H})$ is $(n-1)$-connected and *∞-generating* if $\mathcal{NC}(G, \mathscr{H})$ is contractible.

Abels and Holz proved the following connections between higher generation and algebraic properties of the group.

Proposition 8.15. ([AH93, Theorem 2.4])
Let $\mathscr{H} = \{H_j \mid j \in J\}$ be a family of subgroups of a group G.

(a) $\mathscr{H}$ is 1-generating if and only if $\bigcup_{j \in J} H_j$ generates G.

(b) $\mathscr{H}$ is 2-generating if and only if the natural map $\coprod_{\cap} H_j \to G$ is an isomorphism, where $\coprod_{\cap} H_j$ denotes the free product of $\mathscr{H}$ amalgamated along their intersections.

The coset poset $\mathscr{C}(G)$ of a finite group G is not contractible. Thus $\mathcal{NC}(G, \mathscr{H}_{\mathscr{C}})$ is not contractible for all finite groups G. Hence $\mathscr{H}$ with $\mathscr{H}_{\max} \subseteq \mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}}$ is always finitely-generating for all finite groups.

Now we study sets of subgroups of finitely generated infinite groups. There exist many groups with ∞-generating sets of subgroups. To prove this, we state a general version of the Cone Argument 6.4, the Exchange Argument 8.2, and the Intersection Argument 8.4.

Corollary 8.16. (General Cone Argument)
Let G be a finitely generated infinite group and $\mathscr{H} \subseteq \mathscr{H}_{\text{fi}}$. If $\mathscr{H}_{coH,\max,\text{fi}} \cap \mathscr{H}$ is infinite for all $H \in \mathscr{H}_{\text{fi}}$, then the nerve complex $\mathcal{NC}(G, \mathscr{H})$ is contractible.

Proof. Let U be a finite subcomplex of $\mathcal{NC}(G, \mathscr{H})$ and $H_U := \bigcap_{Hg \in V(U)} H$. By assumption, $\mathscr{H}_{coH_U,\max,\text{fi}}(G) \cap \mathscr{H}$ is infinite. Thus there exists a maximal subgroup $M \in \mathscr{H}$ such that $H_U M = G$. Hence $U * \{M\}$ is a cone in $\mathcal{NC}(G, \mathscr{H})$. □

Corollary 8.17. (General Exchange Argument)
Let G be a finitely generated group and $\mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}}$ such that the following holds.

(i) The nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}} \cap \mathscr{H})$ is contractible.

(ii) $\mathscr{H}_{K,\text{fi}} \cap \mathscr{H}$ is infinite for each subgroup $K \in \mathscr{H}$ of infinite index.

Then the nerve complex $\mathcal{NC}(G, \mathscr{H})$ is contractible.

Proof. Let U be a finite subcomplex of $\mathcal{NC}(G, \mathscr{H})$. As in the Exchange Argument, we can replace each vertex $Kg \in V(U)$ with K an infinite index subgroup by a vertex Hg with $H \in \mathscr{H}_{K,\text{fi}} \cap \mathscr{H}$. Following the proof of the Exchange Argument we are done. □

Corollary 8.18. (General Intersection Argument)
Let G be a finitely generated group and $\mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}}$ such that the following holds.

(i) $\mathscr{H}_{H,\text{fi}} \cap \mathscr{H}$ or $\mathscr{H}_{coH,\max,\text{fi}} \cap \mathscr{H}$ is infinite for each subgroup $H \in \mathscr{H}$.

(ii) For every finite collection of subgroups $\{H_1, ..., H_n\} \subseteq \mathscr{H}$ the intersection $\bigcap_{i=1}^{n} \mathscr{H}_{coH_i,\max,\text{fi}} \cap \mathscr{H}$ is infinite if $\mathscr{H}_{H_i,\text{fi}} \cap \mathscr{H}$ is finite for $i = 1, ..., n$.

Then the nerve complex $\mathcal{NC}(G, \mathscr{H})$ is contractible.

Proof. Let U be a finite subcomplex of $\mathcal{NC}(G, \mathscr{H})$. Using (ii) of the General Exchange Argument we can replace each vertex $Kg \in V(U)$ for which $\mathscr{H}_{K,\text{fi}} \cap \mathscr{H}$ is infinite by a vertex Hg with $H \in \mathscr{H}_{K,\text{fi}} \cap \mathscr{H}$. After these exchanges the new subcomplex U' consists only of vertices Hg such that $\mathscr{H}_{H,\text{fi}} \cap \mathscr{H}$ is finite. Thus by (i) and (ii) there exists a subgroup $M \in \mathscr{H}$ such that $U' * \{M\}$ is a cone in $\mathcal{NC}(G, \mathscr{H})$. □

Example 8.19. (∞-generating sets of subgroups of finitely generated infinite groups)

- A set $\mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}}$ is ∞-generating if $\mathscr{H}_{\text{nor},\max} \subseteq \mathscr{H}$ and $\mathscr{H}_{\text{nor},\max}$ is infinite. The General Intersection Argument holds for such subsets $\mathscr{H}$. Therefore $\mathscr{H}$ can be $\mathscr{H}_{\text{nor},\max}$, $\mathscr{H}_{\mathbb{P}}$, $\mathscr{H}_{\mathbb{P}^n,\max}$, $\mathscr{H}_{\mathbb{P}^n}$, $\mathscr{H}_{\max,\text{fi}}$, $\mathscr{H}_{\text{fi}}$, $\mathscr{H}_{\max}$, or $\mathscr{H}_{\mathscr{C}}$. For the list of such groups see Section 6.2.1.

- $\mathscr{H}_{\mathbb{P}}$, $\mathscr{H}_{\mathbb{P}^n,\max}$, $\mathscr{H}_{\mathbb{P}^n}$, $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\max}$, $\mathscr{H}_{\text{fi}}$, and $\mathscr{H}_{\mathscr{C}}$, are ∞-generating for $\mathbb{Z} \rtimes F$.

- A set $\mathscr{H} \subseteq \mathscr{H}_{\text{fi}}$ is ∞-generating if $\mathscr{H}_{\mathbb{P}}(G) \subseteq \mathscr{H}$ and $\mathscr{H}_{\mathbb{P}}$ is infinite. This is due to the fact that for such sets $\mathscr{H}$ the General Cone Argument holds. Therefore $\mathscr{H}$ can be $\mathscr{H}_{\mathbb{P}}$, $\mathscr{H}_{\mathbb{P}^n,\max}$, $\mathscr{H}_{\mathbb{P}^n}$, $\mathscr{H}_{\max,\text{fi}}$, or $\mathscr{H}_{\text{fi}}$. For the list of such groups see Section 6.2.2.

- A set $\mathscr{H} \subseteq \mathscr{H}_{\text{fi}}$ is ∞-generating if $\mathscr{H}_{\mathbb{P}^n,\max} \subseteq \mathscr{H}$ and $\mathscr{H}_{\mathbb{P}^n,\max}$ is infinite. Again the General Cone Argument holds. Therefore $\mathscr{H}$ can be $\mathscr{H}_{\mathbb{P}^n,\max}$, $\mathscr{H}_{\mathbb{P}^n}$, $\mathscr{H}_{\max,\text{fi}}$, or $\mathscr{H}_{\text{fi}}$. For the list of such groups see Section 6.2.3.

- $\mathscr{H}_k$, $\mathscr{H}_{\max,\text{fi}}$, and $\mathscr{H}_{\text{fi}}$ is ∞-generating if $\mathscr{H}_{coH,\max,\text{fi}}$ is infinite for all $H \in \mathscr{H}_{\text{fi}}$, see Appendix A. For the list of such groups see Section 6.

With the following corollaries we even gather more information about the homotopy among nerve complexes.

Corollary 8.20.
Let G be a finitely generated group, N a normal subgroup of G, and $N \subseteq \Phi_{\text{fi}}(G)$. Suppose that $\mathscr{H}_{N,\text{fi}} \subseteq \mathscr{H} \subseteq \mathscr{H}_{\text{fi}} \cup \{H \in \mathscr{H}_{\mathscr{C}} \mid H \leq N\}$. Then the nerve complexes $\mathcal{NC}(G,\mathscr{H})$ and $\mathcal{NC}(G/N,\mathscr{H}_{\text{fi}})$ are homotopy equivalent. Moreover, if N has finite index, then $\mathcal{NC}(G,\mathscr{H})$ is homotopy equivalent to $\mathcal{NC}(G/N,\mathscr{H}_{\max,\text{fi}})$.

Proof. The first part follows directly using the map r from the proof of Theorem 7.3. If N has finite index, then G/N is a finite group and $\mathcal{NC}(G/N,\mathscr{H}_{\text{fi}})$ and $\mathcal{NC}(G/N,\mathscr{H}_{\max,\text{fi}})$ are homotopy equivalent. □

Corollary 8.21.
Let G be a finitely generated group, N a normal subgroup of G, and $N \subseteq \Phi(G)$. Suppose that $\mathscr{H}_N \subseteq \mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}}$. Then the nerve complexes $\mathcal{NC}(G,\mathscr{H})$ and $\mathcal{NC}(G/N,\mathscr{H}_{\mathscr{C}})$ are homotopy equivalent. Moreover, if N has finite index, then $\mathcal{NC}(G,\mathscr{H}) \simeq \mathcal{NC}(G/N,\mathscr{H}_{\max})$.

Proof. The first part follows directly. Suppose that N has finite index. Then G/N is a finite group and $\mathcal{NC}(G/N,\mathscr{H}_{\mathscr{C}}) \simeq \mathcal{NC}(G/N,\mathscr{H}_{\max})$. □

Finally, we state results for an arbitrary normal subgroup. We define for a finitely generated group G and a subgroup H

$$\mathscr{H}_{coH,\text{fi}}(G) := \{K \in \mathscr{H}_{\text{fi}} \mid HK = G\} \tag{8.4.1}$$

and

$$\mathscr{H}_{coH}(G) := \{K \in \mathscr{H}_{\mathscr{C}} \mid HK = G\}. \tag{8.4.2}$$

Corollary 8.22.
Let G be a finitely generated group and N a normal subgroup of G. With the map of Theorem 7.3, we can prove that $\mathcal{NC}(G,\mathscr{H}) \simeq \mathcal{NC}(G/N,\mathscr{H}_{\text{fi}})$ for N a normal subgroup of G and $\mathscr{H}_{N,\text{fi}} \subseteq \mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}} \setminus \mathscr{C}_{coN,\text{fi}}$.

Corollary 8.23.
Let G be a finitely generated group and N a normal subgroup of G. With the map of Theorem 8.8, we can prove that $\mathcal{NC}(G,\mathscr{H}) \simeq \mathcal{NC}(G/N,\mathscr{H}_{\mathscr{C}})$ for N a normal subgroup of G and $\mathscr{H}_N \subseteq \mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}} \setminus \mathscr{C}_{coN}$.

Now we consider sets of subgroups of finitely generated infinite groups which are finitely-generated.

Example 8.24. (Finitely-generating set)
If a set $\mathscr{H}$ is n-generating it is also m-generating for all $1 \leq m \leq n$. In this example n-generating means that $\mathscr{H}$ is n-generating but not $(n+1)$-generating.

- $\mathscr{H}$ with $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\max} \subseteq \mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}}$ is 2-generating for the first Grigorchuk group. For the proof see Corollary 8.9.
- $\mathscr{H}$ with $\mathscr{H}_{\max,\text{fi}} = \mathscr{H}_{\max} \subseteq \mathscr{H} \subseteq \mathscr{H}_{\mathscr{C}}$ is 1-generating for the Gupta-Sidki p-groups for p an odd prime.
- $\mathscr{H}_{\mathscr{C}}$ and $\mathscr{H}_{\max}$ are 1-generating for a Tarski monster group. In contrast, $\mathscr{H}_{\text{fi}}$ and $\mathscr{H}_{\max,\text{fi}}$ are empty.
- $\mathscr{H}_{\max,\text{fi}}$ and $\mathscr{H}_{\text{fi}}$ are finitely-generating if $\mathscr{H}_{\max,\text{fi}}$ is finite and generates the group.
- $\mathscr{H}_{\max,\text{fi}}$ and $\mathscr{H}_{\text{fi}}$ are n-generating for $S \times (\mathbb{Z}_p)^{n+1}$, where S is an infinite simple group and $n \geq 1$.
- $\mathscr{H}_{\mathscr{C}}$ is n-generating for $G \times (\mathbb{Z}_2)^{n-1}$, where G is a Tarski monster group and $n \in \mathbb{N}$.

Appendix

In the appendix we add results which did not fit in the main part but are interesting by themselves. In fact, some of them are used in the main part.

We start with an examination of the connectivity of posets related to the finite index coset poset of finitely generated infinite groups in Appendix A. Appendix B gives examples of the usage of our theory of subgroup graphs. In Appendix C we study infinite triangle groups and related Coxeter groups.

A Subgroups of index at least k

In this section we consider the posets $\mathscr{C}_k(G)$, which contain all cosets Hg of $\mathscr{C}_{\text{fi}}(G)$ such that H is a subgroup of index at least k. We prove that $\mathscr{C}_k(G)$ and $\mathscr{C}_k(H)$ are contractible for $H \in \mathscr{H}_{\text{fi}}(G)$ if $\mathscr{H}_{coK,\max}(G)$ is infinite for each $K \in \mathscr{H}_{\text{fi}}(G)$, see Propositions A.2 and A.3.

For a finitely generated group G we define

$$\mathscr{C}_k(G) := \{Hg \mid H \in \mathscr{H}_k, g \in G\} \tag{A.0.3}$$

with

$$\mathscr{H}_k(G) := \{H \in \mathscr{H}_{\text{fi}} \mid k < [G : H] < \infty\}. \tag{A.0.4}$$

In fact, $\mathscr{H}_1(G) = \mathscr{H}_{\text{fi}}(G)$ and $\mathscr{C}_1(G) = \mathscr{C}_{\text{fi}}(G)$. The nerve complex $\mathcal{NC}(G, \mathscr{H}_k)$ is a full subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_k(G)$ is a full subcomplex of $\Delta\mathscr{C}_{\text{fi}}(G)$. Since G is finitely generated, $\mathscr{H}_{\text{fi}} \setminus \mathscr{H}_k$ is finite for $k > 1$. Thus $\mathcal{NC}(G, \mathscr{H}_k)$ and $\Delta\mathscr{C}_k(G)$ are infinite simplicial complexes if and only if $\mathscr{H}_{\text{fi}}(G)$ is infinite. But $\Delta\mathscr{C}_k(G)$ might has finite dimension.

Lemma A.1.
Let G be a finitely generated group. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_k)$ and the order complex $\Delta\mathscr{C}_k(G)$ are homotopy equivalent.

Proof. Since $[G : H \cap H'] \geq [G : H], [G : H']$, the set $\mathscr{H}_k(G)$ is closed under intersection. Hence the nerve complex $\mathcal{NC}(G, \mathscr{H}_k)$ and the order complex $\Delta\mathscr{C}_k(G)$ are homotopy equivalent. □

Proposition A.2.
Let G be a finitely generated group such that $\mathscr{H}_{coH,\max,\text{fi}}(G)$ is infinite for all $H \in \mathscr{H}_{\text{fi}}(G)$. Then the nerve complex $\mathcal{NC}(G, \mathscr{H}_k)$ and the order complex $\Delta\mathscr{C}_k(G)$ are contractible for all $k \in \mathbb{N}_{>0}$.

Proof. $\mathscr{H}_{\text{fi}} \setminus \mathscr{H}_k$ is finite for all $k \in \mathbb{N}$. Since the set of maximal subgroups $\mathscr{H}_{coH,\max,\text{fi}}$ is infinite for each $H \in \mathscr{H}_{\text{fi}}$, the set $\mathscr{H}_{coH,\max,\text{fi}} \cap \mathscr{H}_k$ is infinite for each $H \in \mathscr{H}_k$. Analogous to the proof of the Cone Argument 6.4, for every finite subcomplex U of $\mathcal{NC}(G, \mathscr{H}_k)$, there exists a subgroup $M \in \mathscr{H}_{coH,\max,\text{fi}} \cap \mathscr{H}_k$ such that $U * \{M\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_k)$. Thus the nerve complex $\mathcal{NC}(G, \mathscr{H}_k)$ is contractible. □

In particular, for all groups with infinite $\mathscr{H}_{\text{nor},\max}$, $\mathscr{H}_{\mathbb{P}}$, or $\mathscr{H}_{\mathbb{P}^n}$ the nerve complex $\mathcal{NC}(G, \mathscr{H}_k)$ and the order complex $\Delta\mathscr{C}_k(G)$ are contractible.

As in Theorem 6.31 we prove that a finite index subgroup inherits the contractibility.

Proposition A.3.
Let G be a finitely generated group with $\mathscr{H}_{coK,\max,\text{fi}}(G)$ infinite for all $K \in \mathscr{H}_{\text{fi}}(G)$. Let H be a finite index subgroup of G. Then the nerve complex $\mathcal{NC}(H, \mathscr{H}_k)$ and the order complex $\Delta\mathscr{C}_k(H)$ are contractible for all $k \in \mathbb{N}_{>0}$.

Proof. Let U be a finite subcomplex of $\mathcal{NC}(H, \mathscr{H}_k)$. Thus U is a finite subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. By the proof of Proposition A.2, for all $l \in \mathbb{N}_{>0}$, there exists a maximal, finite index subgroup $M \in \mathscr{H}_l(G)$ such that $U * \{M\}$ is a subcomplex of $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$. Thus we can choose $l = kn$ where $n = [G : H]$. Let $m = [G : M]$, $s = [M : M \cap H]$, and $t = [H : M \cap H]$. Then $ms = nt$. Since $m > l = kn$ and $s > 0$, we get $nt = ms > kns$. Thus $t > ks$. Therefore $M \cap H \in \mathscr{H}_k(H)$. Furthermore, $M \cap H \neq H$, since $m > n$. Hence $\mathcal{NC}(H, \mathscr{H}_k)$ is contractible. □

Let Δ denote the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\text{fi}})$ or the order complex $\Delta\mathscr{C}_{\text{fi}}(G)$ and let Δ_k denote the subcomplex $\mathcal{NC}(G, \mathscr{H}_k)$ or $\Delta\mathscr{C}_k(G)$, respectively. Since Δ and Δ_k are contractible, the quotient space Δ/Δ_k is contractible. Consider the group action of G on Δ/Δ_k by right multiplication. Then the orbit space is finite and thus compact. Therefore G acts cocompactly on the finite simplicial complex Δ/Δ_k.

B Subgroup graphs

In this section we give more examples for the usage of our theory of subgroup graphs. First, we describe how to construct subgroup graphs using relators. With this we determine finite index subgroups of semidirect products with $\mathbb{Z}$. Among others we prove that $\mathscr{H}_{\mathbb{P}}(\mathbb{Z} \rtimes G)$ is infinite, which we used in Sections 6.2.2 and 8.1.

We study the direct product of two finitely generated groups $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_2 = \langle X_2 \mid R_2 \rangle$. Hence $G_1 \times G_2 = \langle X_1, X_2 \mid R_1, R_2, R_3 \rangle$. Thus we know that the subgroup graph $\Gamma(H)$ of a finite index subgroup $H < G_1 \times G_2$ fulfills the relators R_1, R_2 and R_3, is finite, connected, and $(X_1 \cup X_2)$-regular. Therefore each connected component of the graph $\Gamma(H)|_{X_i}$ fulfills the relators R_i, is finite, and X_i-regular for $i = 1, 2$. Thus each component of $\Gamma(H)|_{X_i}$ is a subgroup graph of a finite index subgroup in G_i. Therefore, if we create a subgroup graph $\Gamma(H)$ of a finite index subgroup $H < G_1 \times G_2$ we construct them using the subgroup graphs $\Gamma(H_i)$ of the finite index subgroups $H_i < G_i$ and glue them together such that R_3 is fulfilled. This decreases the number of possible subgroup graphs. This observation is also true for the free product and semidirect products with different R_3. Moreover, we can generalize it to the following observation.

Let $G = \langle X \mid R \rangle$ with X and R being finite and let $r \in R$ be free and cyclically reduced. We create a reduced path with $\mu(p) = r$ and $o(p) = t(p)$. We call this X-graph Γ_r. It need not be X-regular, but since p is reduced it is folded. We create even more folded X-graphs Γ, such that there is an X-graph morphism $\pi \colon \Gamma_r \to \Gamma$. If we do this for each $r \in R$, these graphs are the only elements a subgroup graph can be built of. For example take $r = a^6$. Then Γ_r is an $(a, 6)$-circle. Except for Γ_r itself the graphs Γ are an a-loop, an $(a, 2)$-circle, and an $(a, 3)$-circle. Thus we know that only these circles with label a are possible in a subgroup graph of a finite index subgroup of a group, where a has order 6.

Let us study the groups $\mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{Z} \rtimes \mathbb{Z}_2$ with this information. Although the subgroups of these groups are well known, studying them leads to information about $\mathbb{Z} \rtimes G$.

B.1 Finite index subgroups of $\mathbb{Z} \times \mathbb{Z}_2$

First, we consider the direct product of $\mathbb{Z} = \langle a \mid \varnothing \rangle$ and $\mathbb{Z}_2 = \langle b \mid b^2 \rangle$. Thus we know that for a there is any (a, n)-circle possible with $n \in \mathbb{N}_{>0}$. For b we only have b-loops or $(b, 2)$-circles. Moreover, we have the graphs Γ which we get from $bab^{-1}a^{-1}$. We will not number them here, but they give the information that, if there is a b-loop, then each edge, labeled b, is a loop. Each subgroup graph is then a finite graph consisting of the just numbered graphs glued together. Therefore we get the subgroup graphs of Figure 24.

From now on we draw an undirected edge, labeled x, between two vertices u and v, meaning an edge, labeled x, from u to v and one from v to u. Thus an undirected edge, labeled x, is an $(x, 2)$-circle and an undirected loop is just a directed loop.

Each graph of Figure 24 provides only one subgroup. Therefore the finite index subgroups of $\mathbb{Z} \times \mathbb{Z}_2$ are as follows:

- Γ_n provides: $\langle a^n, b \rangle$,
- Γ'_{2n} provides: $\langle a^n \rangle$,
- Γ''_{2n} provides: $\langle a^{2n}, a^n b \rangle$.

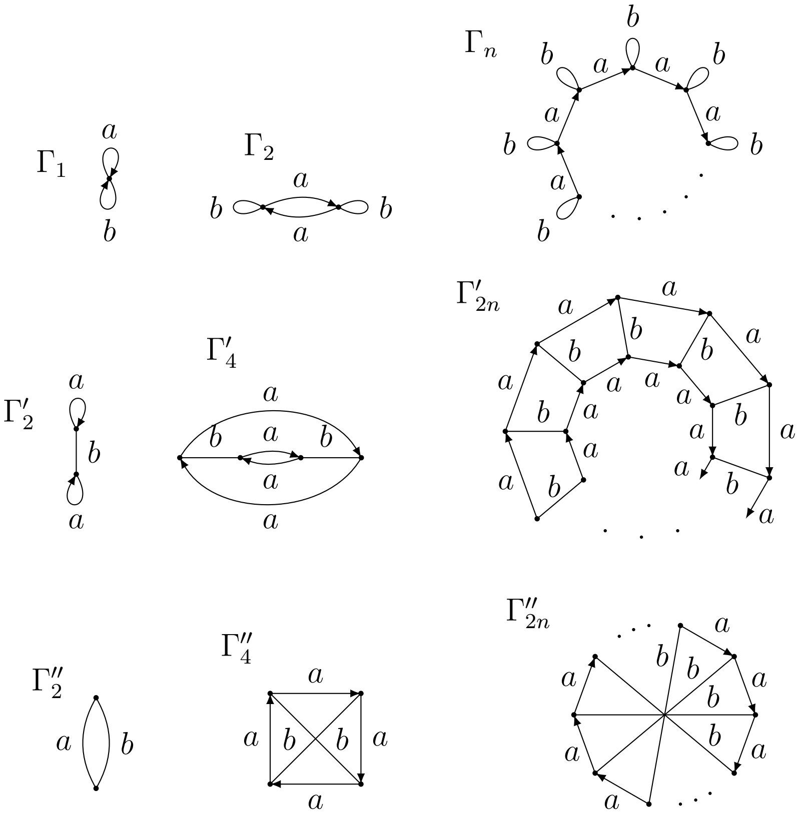

Figure 24: Subgroup graphs of all finite index subgroups of $\mathbb{Z} \times \mathbb{Z}_2$.

B.2 Finite index subgroups of $\mathbb{Z} \rtimes \mathbb{Z}_2$

Now we consider all proper finite index subgroups of the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2 = \langle\, a, b \,|\, b^2, bab^{-1}a \,\rangle$. We find the corresponding subgroup graphs in the Figures 25 and 26. We know the components from which the subgroup graph can be built of. For a and b they are as for the direct product, but the relator $bab^{-1}a$ changes the graphs. This time a graph can have both b-loops and $(b, 2)$-circles. We construct them as follows, proving also that there does not exist any other finite index subgroup.

We start with a b-loop at a vertex of an (a, n)-circle. If $n = 1$, then we get $\Gamma(\mathbb{Z} \rtimes \mathbb{Z}_2)$. If n is odd, then we end up with Γ_n. If n is even, then we end up with Γ'_n.

If we start with an (a, n)-circle and a $(b, 2)$-circle, such that they share two vertices, then we have the following possibilities. The $(b, 2)$-loop connects two adjacent vertices. Then we end up with Γ_n, if n is odd and with Γ''_n if n is even. The $(b, 2)$-circle connects two non-adjacent vertices and p, p' are the reduced paths, with label a^m and $a^{m'}$, for $1 \leq m \leq m' < n$, between those two vertices. If n, m and m' are even, then we get Γ'_n. If n is even and m and m' are odd, then we get Γ''_n. If n is odd, then we get Γ_n.

If we start with a $(b, 2)$-circle sharing only one vertex with an (a, n)-circle, then we end up with Γ'''_{2n}.

Finally, if we start with an a-loop but no b-loop, then we get Γ'''_2.

Using subgroup graphs and the relation $ba = a^{-1}b$ we get the following proper finite index subgroups.

The graphs Γ_{2n+1} with $2n + 1$ vertices for $n \in \mathbb{N}_{>0}$ are depicted in Figure 25. Each vertex of Γ_{2n+1} provides a different language and thus a different subgroup. Starting with the vertex with the b-loop, we have the following groups.

- Γ_3 provides: $\langle a^3, b\rangle$, $\langle a^3, ab\rangle$, $\langle a^3, a^2b\rangle$.
- Γ_5 provides: $\langle a^5, b\rangle$, $\langle a^5, a^3b\rangle$, $\langle a^5, ab\rangle$, $\langle a^5, a^4b\rangle$, $\langle a^5, a^2b\rangle$.
- Γ_{2n+1} provides: $\langle a^{2n+1}, b\rangle$, $\langle a^{2n+1}, a^{2n-1}b\rangle$, $\langle a^{2n+1}, a^{2n-3}b\rangle$, ..., $\langle a^{2n+1}, a^3b\rangle$, $\langle a^{2n+1}, ab\rangle$, $\langle a^{2n+1}, a^{2n}b\rangle$, $\langle a^{2n+1}, a^{2n-2}b\rangle$, ..., $\langle a^{2n+1}, a^4b\rangle$, $\langle a^{2n+1}, a^2b\rangle$.

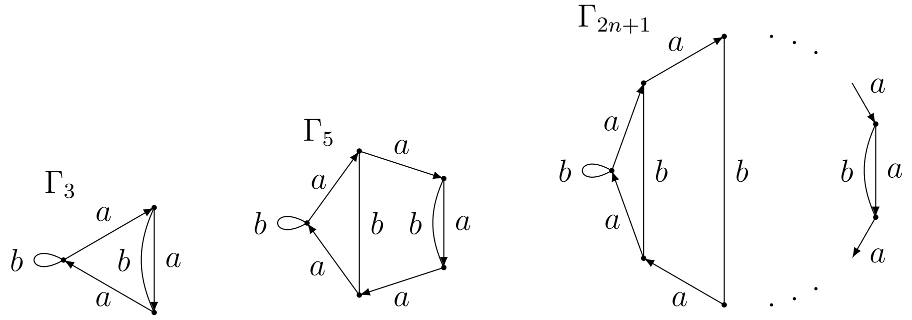

Figure 25: Subgroup graphs of all proper subgroups of odd index in the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2 = \langle\, a, b \,|\, b^2, bab^{-1}a \,\rangle$.

Figure 26 pictures all graphs with an even number of vertices. The graphs Γ'_{2n} and Γ''_{2n} have both $2n$ vertices with $n \in \mathbb{N}_{>0}$. Both graphs have a rotational symmetry of order 2. Thus each graph provides only n different subgroups. For Γ'_{2n} we start at the left vertex with a b-loop.

- Γ'_2 provides: $\langle a^2, b\rangle$.
- Γ'_4 provides: $\langle a^4, b\rangle$, $\langle a^4, a^2b\rangle$.
- Γ'_{2n} provides: $\langle a^{2n}, b\rangle$, $\langle a^{2n}, a^{2n-2}b\rangle$, $\langle a^{2n}, a^{2n-4}b\rangle$, ..., $\langle a^{2n}, a^4b\rangle$, $\langle a^{2n}, a^2b\rangle$.

For Γ''_{2n} we start with the upper left vertex of the b-edge connecting two adjacent vertices.

- Γ''_2 provides: $\langle a^2, ab\rangle$.

- Γ''_4 provides: $\langle a^4, a^3b\rangle$, $\langle a^4, ab\rangle$.
- Γ''_{2n} provides: $\langle a^{2n}, a^{2n-1}b\rangle$, $\langle a^{2n}, a^{2n-3}b\rangle$, ...,$\langle a^{2n}, a^3b\rangle$, $\langle a^{2n}, ab\rangle$.

The graph Γ'''_{2n} has $2n$ vertices with $n \in \mathbb{N}_{>0}$. The languages of the graph Γ'''_{2n} are equal for each vertex.

- Γ'''_{2n} provides $\langle a^n\rangle$.

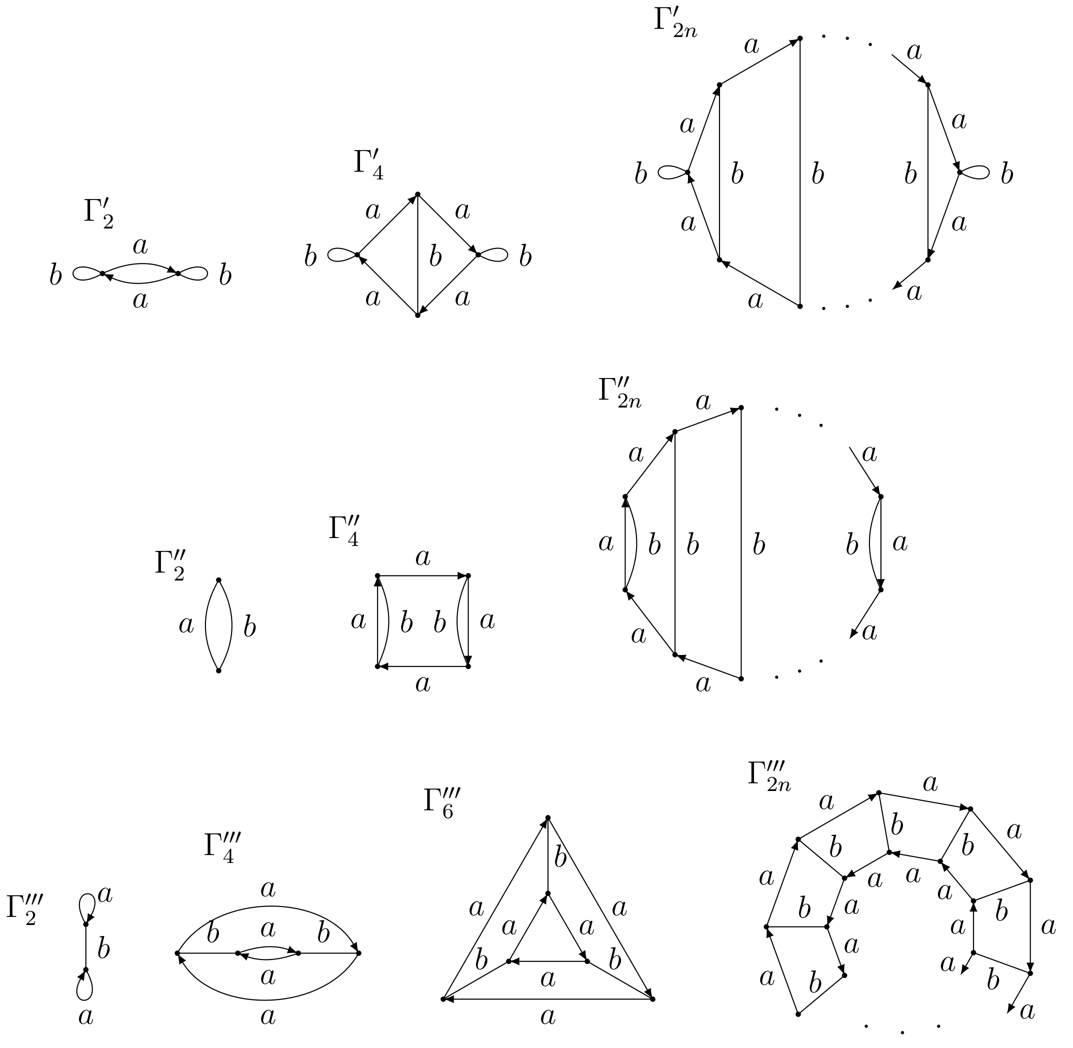

Figure 26: Subgroup graphs of all proper subgroups of even index in $\mathbb{Z} \rtimes \mathbb{Z}_2$.

One can see that $\mathbb{Z} \rtimes \mathbb{Z}_2$ has a lot more subgroups of finite index than $\mathbb{Z} \times \mathbb{Z}_2$. The direct product $\mathbb{Z} \times \mathbb{Z}_2$ has only one subgroup of index n if n is odd, and 3 of index n if n is even. The semidirect product has n subgroups of index n if n is odd, and $n + 1$ if n is even.

B.3 $\mathbb{Z} \rtimes_\psi F$ for F being a finite group

We now prove results we used in Section 8.1. Let $F = \langle Y\rangle$, $\mathbb{Z} = \langle a\rangle$ and $\psi\colon F \to \mathrm{Aut}(\mathbb{Z})$ be a homomorphism. We divide F in two types of elements

$F_+ := \{g \in F \mid \psi(g)(a) = a\}$ and $F_- := \{g \in F \mid \psi(g)(a) = a^{-1}\}$. If $g \in F_+$ we denote g by g_+ and if $g \in F_-$ we denote g by g_-. By definition, $(g_-)^2 = (g^2)_+$. If $\psi = \mathrm{id}$, then $\mathbb{Z} \rtimes F$ is the direct product and $F_+ = F$.

Lemma B.1.
Let F be a finite group. Then every subgroup of $\mathbb{Z} \rtimes_\psi F$ is finitely generated.

Proof. Consider $\mathbb{Z} \rtimes_\psi F = \langle\, a, Y \mid yay^{-1}(\psi(y)(a))^{-1}, y \in Y \,\rangle$. Then an element is of the form $a^s g$ with $g \in F$ and s an integer.

Suppose that H is an infinitely generated subgroup. Since F is finite, there exists at least one $g \in F$ with $\mathrm{ord}(g) = n$ such that $a^s g, a^t g \in H$ with $0 \neq s \neq t$.

Assume that $g \in F_+$. Since $g_+ a = a g_+$, we have $(a^s g_+)^n = a^{ns}$. Thus $\langle a^{ns} \rangle \leq H$ and H has finite index.

Assume that $g \in F_-$. Since $g_- a = a^{-1} g_-$, we get

$$(a^s g_-)^2 = a^s g_- a^s g_- = a^s a^{-s} (g_-)^2 = (g^2)_+$$

and thus

$$(a^s g_-)^n = \begin{cases} (g_-)^n = (g^n)_+, & \text{for } n \text{ even,} \\ a^s (g^n)_-, & \text{for } n \text{ odd.} \end{cases}$$

Since g_- has always even order, we get

$$a^s g_- (a^t g_-)^{n-1} = a^s g_- a^t (g_-)^{n-1} = a^{s-t} (g_-)^n = a^{s-t} \in H.$$

□

We define $Y_+ := Y \cap F_+$ and $Y_- := Y \cap F_-$. Then $a^s Y_- = \{a^s y_- \mid y_- \in Y_-\}$. If $\psi = \mathrm{id}$, then $Y_- = \varnothing$.

Lemma B.2.
Let H be a finitely generated subgroup of infinite index in $\mathbb{Z} \rtimes_\psi F$. Then H is a subgroup of $H_s := \langle a^s Y_-, Y_+ \rangle = \{a^s g_-, g_+ \mid g_+ \in F_+, g_- \in F_-\}$ for an $s \in \mathbb{N}_{>0}$.

Proof. The case of the direct product is included, since $H_s = F$ and each subgroup of infinite index is a subgroup of F.

For $\psi \neq \mathrm{id}$ we prove that $H_s = \{a^s g_-, g_+ \mid g_+ \in F_+, g_- \in F_-\}$. Consider $h := a^{s_1} y_1 \cdots a^{s_n} y_n$ with $s_i = 0$ if $y_i \in F_+$ and $s_i = s$ if $y_i \in F_-$. If the number of $y_i \in F_-$ is odd, then $h = a^s y_1 \cdots y_n = a^s g$ and $g \in F_-$. If the number of $y_i \in F_-$ is even, then $h = y_1 \cdots y_n$ and $h \in F_+$. Therefore the equation holds. Thus H_s has $|F|$ elements and infinite index in $\mathbb{Z} \rtimes F$.

By the previous proof, we know the following about an infinite index subgroup H. If $a^s g_- \in H$, then $a^t g_- \notin H$ for all $t \neq s$ and $a^s g_+, a^s \notin H$ for $s \neq 0$. Thus the only possible elements are of the form $a^s g_-$ and g_+. Hence $H \leq H_s$ for some $s \neq 0$. □

Proposition B.3.
Let F be a finite group. Then the nerve complex $\mathcal{NC}(\mathbb{Z} \rtimes_\psi F, \mathscr{H}_\mathscr{C})$ and the order complex $\Delta\mathscr{C}(\mathbb{Z} \rtimes_\psi F)$ are contractible.

Proof. Let $H \in \mathscr{H}_{\mathscr{C}} \setminus \mathscr{H}_{\mathrm{fi}}$. Then $H \leq \langle a^s Y_-, Y_+ \rangle < \langle a^t, a^s Y_-, Y_+ \rangle$ for each $t \in \mathbb{N}$ with $t > |s|$. Using subgroup graphs we can prove that $\langle a^t, a^s Y_-, Y_+ \rangle$ is a subgroup of index t, see Appendix B.4. For the graphs with odd t see Figure 27. Thus $\mathscr{H}_{H,\mathrm{fi}}(\mathbb{Z} \rtimes_\psi F)$ is infinite for each $H \in \mathscr{H}_{\mathscr{C}} \setminus \mathscr{H}_{\mathrm{fi}}$ and Lemma 8.2 holds. □

Since $\mathscr{H}_{\mathrm{nor,max}}$ is infinite for $F \rtimes \mathbb{Z}$, one might expect that this is also true for $\mathbb{Z} \rtimes F$. But this fails if it is not the direct product.

Proposition B.4.
Let F be a finite non-trivial group and ψ not the identity. Then $\mathscr{H}_{\mathbb{P}}(\mathbb{Z} \rtimes_\psi F)$ is finite for any semidirect product $\mathbb{Z} \rtimes F$.

Proof. Using subgroup graphs we can prove that $\mathscr{H}_{\mathrm{nor,max}}(\mathbb{Z} \rtimes F)$ is finite, if $\mathbb{Z} \rtimes F$ is not the direct product. If N is a normal maximal subgroup in $\mathbb{Z} \rtimes F$, then $\Gamma(N)$ has p vertices and $(\Gamma(N), 1_H) \cong (\Gamma(N), v)$ for all vertices v of $\Gamma(N)$. If $\Gamma(N)|_Y$ is connected, then $\Gamma(N)|_Y$ is the subgroup graph of a proper normal subgroup N' of F. Since F is finite, there are only finitely many subgroups. Since p is prime, $\Gamma(N)|_a$ is either an (a,p) circle or p unconnected a-loops. Thus for each N' there are at most two possible normal subgroups N. If $\Gamma(N)|_Y$ is not connected, then it contains of p copies of $\Gamma(F)$. Thus $\Gamma(N)|_a$ is an (a,p)-circle. Since ψ is not the identity there exists an element $y_- \in Y$. Therefore such a subgroup graph cannot exist. □

B.4 $\mathbb{Z} \rtimes G$ with G being a finitely generated group

After investigating the semidirect product $\mathbb{Z} \rtimes_\psi \mathbb{Z}_2$ it is easier to consider the semidirect product $\mathbb{Z} \rtimes_\psi G$ for a finitely generated group G. Thus we now prove that $\mathscr{H}_{\mathbb{P}}(\mathbb{Z} \rtimes G)$ is infinite, which we stated in Example 6.21, and that $\langle a^t, a^s Y_-, Y_+ \rangle$ is a finite index subgroup for all $0 \neq t \in \mathbb{N}$, which we used in the proof of Proposition B.3.

Proposition B.5.
Let $G = \langle\, Y \mid R_Y \,\rangle$ be a finitely generated group with finite Y. Then $\mathscr{H}_{\mathbb{P}}(\mathbb{Z} \times_\psi G)$ is infinite. Therefore the nerve complex $\mathcal{NC}(\mathbb{Z} \rtimes_\psi G, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(\mathbb{Z} \rtimes_\psi G)$ are contractible.

Proof. Let $\psi\colon G \to \mathrm{Aut}(\mathbb{Z})$ be a chosen homomorphism. We divide Y in disjoint sets $Y_+ = \{y \in Y \mid \psi(y) = id\}$ and $Y_- = \{y \in Y \mid \psi(y)(a) = a^{-1}\}$. Thus $\mathbb{Z} \rtimes_\psi G = \langle\, a, Y \mid R_Y, R_+, R_- \,\rangle$ with relators $R_+ = \{yay^{-1}a^{-1} \mid y \in Y_+\}$ and $R_- = \{yay^{-1}a \mid y \in Y_-\}$. Let Γ_p be as in Figure 27.

Let $r \in R_Y$. Then $r = y_1 \cdots y_n$ with $y_i \in Y^{\pm}$. One connected component of the subgraph $\Gamma_p|_Y$ is the subgroup graph $\Gamma_{Y,R_Y}(G)$. The others are graphs with two vertices connected by a $(y,2)$-circle for each $y \in Y_-$ and a y-loop at both vertices for each $y \in Y_+$. Thus a path p_r with $\mu(p_r)$ has the same origin and terminus, if and only if $l_-(r)$ is even. Since $r = 1$, we have $\psi(r) = \psi(1)$. Thus $l_-(r) = |\{y_i \mid r = y_1 \cdots y_n, y_i \in Y_-^{\pm}\}|$ is even. Therefore each Γ_p fulfills R_Y.

If we change all labels b to y in Γ_p of Figure 24, we get the graph $\Gamma_p|_{\{a,y\}}$ for each $y \in Y_+$. Moreover, $\Gamma_p|_{\{a,y\}}$ is the same graph as Γ_p of Figure 25 and $\Gamma_2|_{\{a,y\}}$

is Γ_2' of Figure 26 with $b = y$ for each $y \in Y_-$. Therefore each Γ_p fulfills R_+ and R_-. Thus $\mathscr{H}_{\mathbb{P}}(Z \rtimes F)$ is infinite. □

Each graph Γ_p of Figure 27 provides p different subgroups if p is an odd prime.

- Γ_2 provides: $\langle a^2, Y \rangle$.
- Γ_3 provides: $\langle a^3, Y_-, Y_+ \rangle$, $\langle a^3, aY_-, Y_+ \rangle$, $\langle a^3, a^2Y_-, Y_+ \rangle$.
- Γ_p provides: $\langle a^p, Y_-, Y_+ \rangle$, $\langle a^p, a^{p-2}Y_-, Y_+ \rangle$, $\langle a^p, a^{p-4}Y_-, Y_+ \rangle$, ..., $\langle a^p, aY_-, Y_+ \rangle$, $\langle a^p, a^{p-1}Y_-, Y_+ \rangle$, $\langle a^p, a^{p-3}Y_-, Y_+ \rangle$, ...,$\langle a^p, a^2Y_-, Y_+ \rangle$.

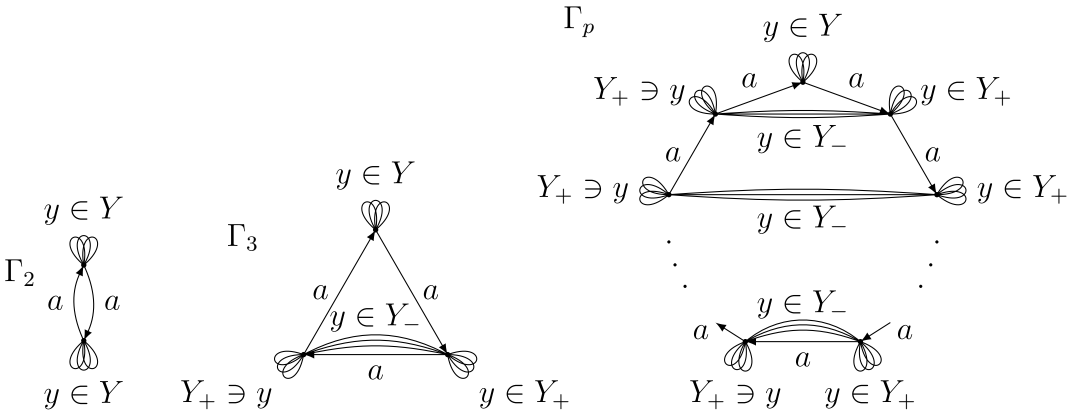

Figure 27: Subgroup graph Γ_p of $\mathbb{Z} \rtimes_\psi G$ with $G = \langle Y \rangle$, $|Y| < \infty$, and $Y = Y_+ \cup Y_-$ with $Y_+ = \{y \in Y \mid \psi(y) = id\}$ and $Y_- = \{y \in Y \mid \psi(y)(a) = a^{-1}\}$.

If we change the graphs of Figure 26 in the following way we get subgroup graphs of even index. Take the graph Γ_{2n}' or Γ_{2n}'' and add a loop, labeled y_+, for each $y_+ \in Y_+$ at each vertex and add an edge for each $y_- \in Y_-$, labeled y_-, such that it connects the same vertices as the edges with label b do. Then delete the edges with label b. The resulting graphs provide the following subgroups.

- Modified Γ_{2n}' provides: $\langle a^{2n}, Y \rangle$, $\langle a^{2n}, a^{2n-2}Y_-, Y_+ \rangle$, $\langle a^{2n}, a^{2n-4}Y_-, Y_+ \rangle$, ..., $\langle a^{2n}, a^4Y_-, Y_+ \rangle$, $\langle a^{2n}, a^2Y_-, Y_+ \rangle$.
- Modified Γ_{2n}'' provides: $\langle a^{2n}, a^{2n-1}Y_-, Y_+ \rangle$, ..., $\langle a^{2n}, a^3Y_-, Y_+ \rangle$, $\langle a^{2n}, aY_-, Y_+ \rangle$.

C Triangle groups and related Coxeter groups

After studying the infinite right angled Coxeter groups we are interested in other special Coxeter groups. Thus we study the infinite triangle groups, focusing on the euclidean triangle groups. In Section C.1 we study hyperbolic triangle groups and prove that the finite index coset poset is contractible for many of them. We used this result in Section 6.2.2. In Section C.2 we consider the euclidean triangle groups. Among this we prove that $\mathscr{H}_{\mathbb{P}^2}$ is infinite for each euclidean triangle group, which we used in Section 6.2.3.

A triangle group is a special Coxeter group with the following presentation $\langle\, a, b, c \,|\, a^2, b^2, c^2, (ab)^l, (bc)^m, (ac)^n \,\rangle$, where $l \leq m \leq n \in \mathbb{N}_{>1} \cup \{\infty\}$. We denote this group with $\Delta(l, m, n)$. Recall that if an exponent is ∞, then there is no relator. Note that in some literature this group is called the full triangle group and the alternating subgroup of index two is called the triangle group. There are three types of triangle groups. If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$, then $\Delta(l, m, n)$ is finite. This is the case if (l, m, n) equals $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, or $(2, 2, n), n > 1$. If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$, the group $\Delta(l, m, n)$ is called euclidean. There are only three euclidean triangle groups, $\Delta(2, 4, 4)$, $\Delta(2, 3, 6)$ and $\Delta(3, 3, 3)$. If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, the group $\Delta(l, m, n)$ is called hyperbolic. The euclidean and hyperbolic triangle groups are infinite.

C.1 Hyperbolic triangle groups

Using the subgroup graphs we prove that the set $\mathscr{H}_{\mathbb{P}}$ is infinite for all triangle groups with $n = \infty$ in Section C.1.1. In Section C.1.2 we consider those with finite exponent. We prove that the finite index coset poset is contractible for many hyperbolic triangle groups and related Coxeter groups.

C.1.1 Containing infinite exponents

In this part we only study triangle groups $\Delta(l, m, n)$ with $n = \infty$. Moreover, we investigate some related Coxeter groups.

Proposition C.1.
Let G be one of the hyperbolic triangle groups $\Delta(\infty, \infty, \infty)$, $\Delta(l, \infty, \infty)$ or $\Delta(l, m, \infty)$ with $l, m \in \mathbb{N}_{>1}$. Then $\mathscr{H}_{\mathbb{P}}$ is infinite. Therefore the nerve complex $\mathcal{NC}(G, \mathscr{H}_{\mathrm{fi}})$ and the order complex $\Delta\mathscr{C}_{\mathrm{fi}}(G)$ are contractible.

Proof. We proved in Section 6.2.2 that $\mathscr{H}_{\mathbb{P}}$ is infinite for $\Delta(\infty, \infty, \infty)$ and $\Delta(2l, 2m, \infty)$ for any $l, m \in \mathbb{N}_{>0}$. Figure 28 proves that $\mathscr{H}_{\mathbb{P}}(\Delta(l, \infty, \infty))$ is infinite for all $l \in \mathbb{N}_{>1}$. Figure 29 shows subgroup graphs of $\Delta(2l, 3m, \infty)$ with $5 + 6k$ vertices. By Dirichlet's Theorem, there are infinitely many primes of the form $5 + 6k$. Thus $\mathscr{H}_{\mathbb{P}}(\Delta(2l, 3m, \infty))$ is infinite for any $l, m \in \mathbb{N}_{>0}$. The subgroup graphs in Figure 30 have $q + 2 + 2qk$ vertices. Since $q > 3$ is odd, $q + 2$ is odd. Thus $2q$ and $q + 2$ are coprime. Therefore there exist infinitely many k such that $q + 2 + 2qk$ is prime. Hence $\mathscr{H}_{\mathbb{P}}(\Delta(2l, qm, \infty))$ is infinite for any prime $q > 3$ and all $l, m \in \mathbb{N}_{>0}$. The subgroup graphs in Figure 31 have $q + (2p - 2 + 2q - 2)k$

vertices. To prove that $\mathscr{H}_{\mathbb{P}}(\Delta(pl, qm, \infty))$ is infinite for any $2 < p \leq q$ with p, q prime and $l, m \in \mathbb{N}_{>0}$ we show that q and $2(p+q-2)$ are coprime. Consider $p = q$. Then q and $4(q-1)$ are coprime, since q is odd. Consider $p < q$. Then $q < p+q-2 < 2q$. Therefore q and $p+q-2$ are coprime. $\square$

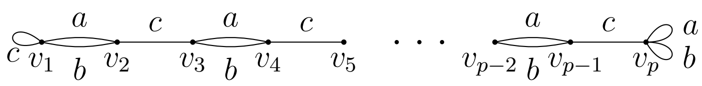

Figure 28: Subgroup graphs with p vertices of subgroups of $\Delta(l, \infty, \infty)$ for p prime.

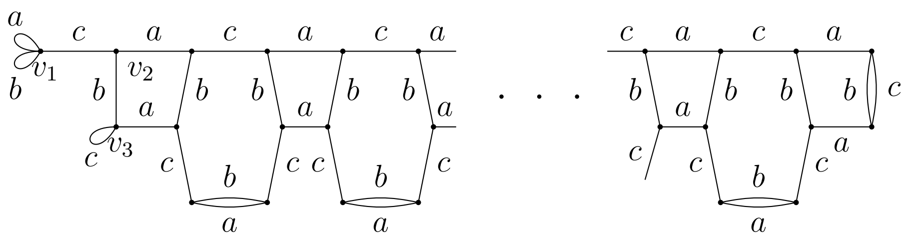

Figure 29: Subgroup graphs with $5+6k$ vertices of subgroups of $\Delta(2l, 3m, \infty)$, where k is the number of hexagons.

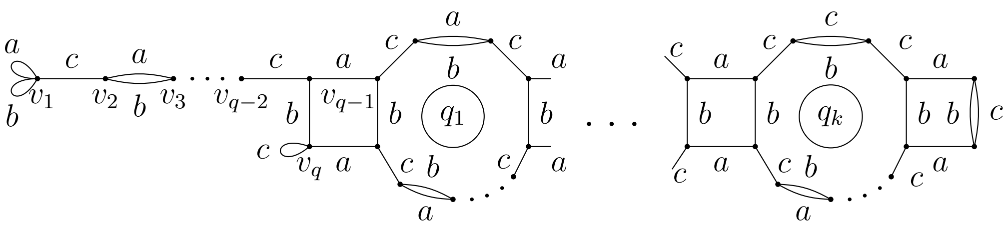

Figure 30: Subgroup graphs with $q+2+2qk$ vertices of subgroups of $\Delta(2l, qm, \infty)$ with $q > 3$ prime.

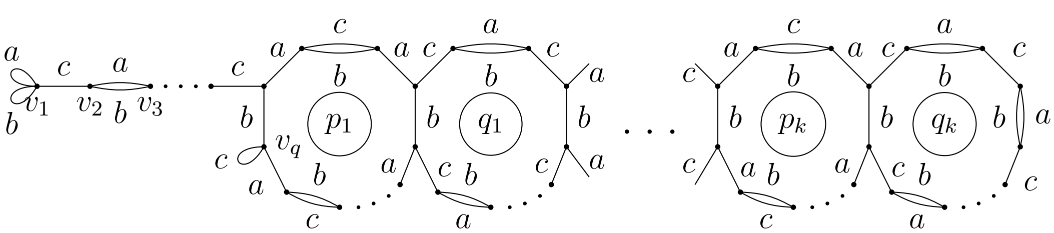

Figure 31: Subgroup graphs with $q + (2p-2+2q-2)k$ vertices of subgroups of $\Delta(pl, qm, \infty)$ with $2 < p \leq q$ and p, q prime.

In Section 6.2 we already studied some Coxeter groups which contain the triangle groups $\Delta(2l, 2m, \infty)$ as subgroups, for example the infinite right angled

Coxeter groups contain $\Delta(2l, 2m, \infty)$ if $l, m \in \{1, \infty\}$. There also exist many Coxeter groups containing $\Delta(l, m, \infty)$ or $\Delta(l, \infty, \infty)$ as subgroups. Our purpose is not to study all of them, but we consider some.

Corollary C.2.
Let $W = \langle s_1, ..., s_n \mid s_i^2, (s_i s_j)^{m_{i,j}}, 1 \leq i < j \leq n \rangle$ be a Coxeter group such that $m_{1,2} = pl$, $m_{2,3} = qm$, and $m_{1,3} = \infty$ for primes $p \leq q$ with $(p, q) \neq (2, 2)$. Suppose that one of the following holds:

(i) $m_{1,i}, m_{2,i}, m_{3,i} \in 2\mathbb{N}_{>0} \cup \{\infty\}$ *for all* $i = 4, ..., n$,

(ii) $m_{1,i} \in p\mathbb{N}_{>0} \cup \{\infty\}$ *and* $m_{3,i} \in q\mathbb{N}_{>0} \cup \{\infty\}$ *for all* $i = 4, ..., n$,

(iii) $m_{2,i} \in p\mathbb{N}_{>0} \cup \{\infty\}$ *and* $m_{3,i} = \infty$ *for all* $i = 4, ..., n$,

(iv) $m_{1,i} = \infty$ *and* $m_{2,i} \in q\mathbb{N}_{>0} \cup \{\infty\}$ *for all* $i = 4, ..., n$.

Then $\mathscr{H}_{\mathbb{P}}(W)$ is infinite. Therefore the nerve complex $\mathcal{NC}(W, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(W)$ are contractible.

Proof. We change the graphs of Figures 29–31 in the following way. First, change each label a to s_1, b to s_2, and c to s_3. Then we proceed in the following way.

(i) In this case we can add a loop, labeled s_i, for each $i = 4, ..., n$ to every vertex of the graphs of Figures 29–31. The resulting graphs fulfill the defining relators of the Coxeter group W and thus provide infinitely many subgroups of prime index.

(ii)–(vi) To the graphs of Figure 29 for $p = 2$ and $q = 3$, Figure 30 for $p = 2$ and $q > 3$, and Figure 31 for $2 < p \leq q$ we add edges in the following way. For each $i = 4, ..., n$ we add an edge, labeled s_i, connecting the same vertices as an edge, labeled x, does. Here $x = b$ in case (ii), $x = a$ in the case (iii), and $x = c$ in the case (iv).

Thus we proved that $\mathscr{H}_{\mathbb{P}}$ is infinite for those Coxeter groups. □

Now we consider Coxeter groups which are related to the triangle groups $\Delta(l, \infty, \infty)$.

Corollary C.3.
Let $W = \langle s_1, ..., s_n \mid s_i^2, (s_i s_j)^{m_{i,j}}, 1 \leq i < j \leq n \rangle$ be a Coxeter group such that $m_{1,2} = l$ and $m_{2,3} = m_{1,3} = \infty$. Suppose that one of the following holds:

(i) $m_{1,i}, m_{2,i}, m_{3,i} \in 2\mathbb{N}_{>0} \cup \{\infty\}$ *for all* $i = 4, ..., n$,

(ii) $m_{3,i} = \infty$ *for all* $i = 4, ..., n$,

(iii) $m_{1,i} = m_{2,i} = \infty$ *for all* $i = 4, ..., n$.

Then $\mathscr{H}_{\mathbb{P}}(W)$ is infinite. Therefore the nerve complex $\mathcal{NC}(W, \mathscr{H}_{\text{fi}})$ and the order complex $\Delta\mathscr{C}_{\text{fi}}(W)$ are contractible.

Proof. We change the graph of Figure 28 in the following way. First, change each label a to s_1, b to s_2, and c to s_3. Then we proceed in the following way.

(i) To any vertex of the graph of Figure 28 we add an loop, labeled s_i, for each $i = 4, ..., n$.

(ii) For each $i = 4, ..., n$ we add an edge, labeled s_i, connecting the same vertices as an edge, labeled a, does. Thus if Γ denotes the resulting graph, then $(\Gamma|_a, v) \cong (\Gamma|_{s_i}, v)$ for $i = 4, ..., n$.

(iii) For each $i = 4, ..., n$ we add an edge, labeled s_i, connecting the same vertices as an edge, labeled c, does.

Thus we proved that $\mathscr{H}_{\mathbb{P}}$ is infinite for those Coxeter groups. □

C.1.2 Only finite exponents

Now we consider hyperbolic triangle groups with finite exponents. By drawing subgroup graphs, we prove that many of them have an infinite set of prime index subgroups and therefore their finite index coset poset is contractible. The groups for which we prove this are $\Delta(2l, 4m, 6n)$, $\Delta(3l, 4m, 5n)$, $\Delta(3l, 4m, 7n)$, $\Delta(3l, 5m, 6n)$, $\Delta(3l, 5m, 10n)$, and $\Delta(4l, 7m, 10n)$. They are somehow randomly chosen and therefore just more examples.

As in the Corollaries C.2 and C.3 the graphs can be used to prove that the finite index coset poset is contractible for related Coxeter groups.

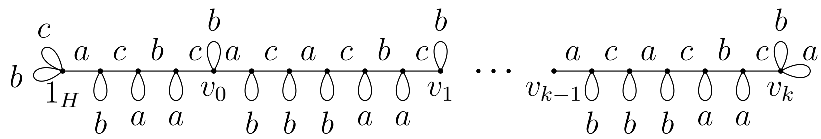

Figure 32: Graphs with $5 + 6k$ vertices, which fulfill the defining relators of the triangle group $\Delta(2l, 4m, 6n)$.

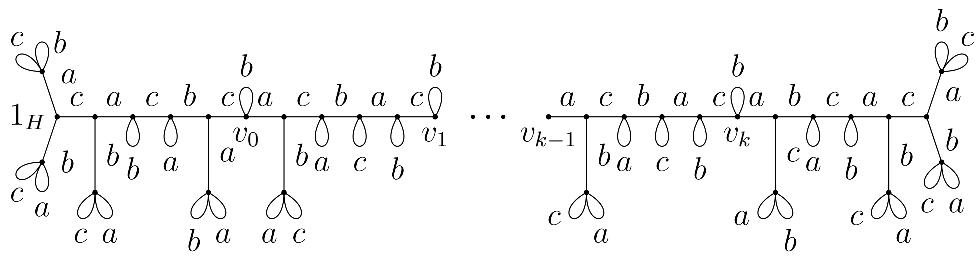

Figure 33: Graphs with $19 + 6k$ vertices, which fulfill the defining relators of the triangle group $\Delta(3l, 4m, 5n)$.

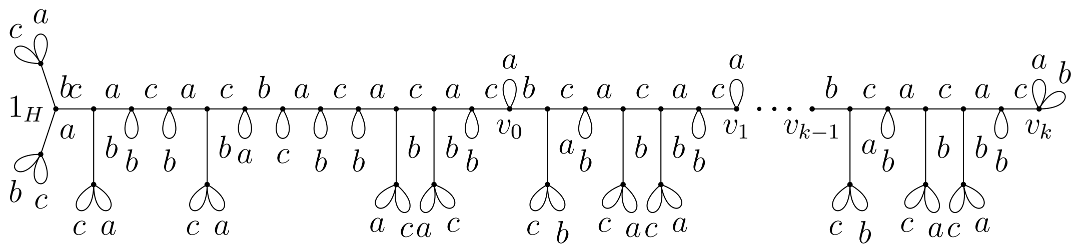

Figure 34: Graphs with $19 + 9k$ vertices, which fulfill the defining relators of the triangle group $\Delta(3l, 4m, 7n)$.

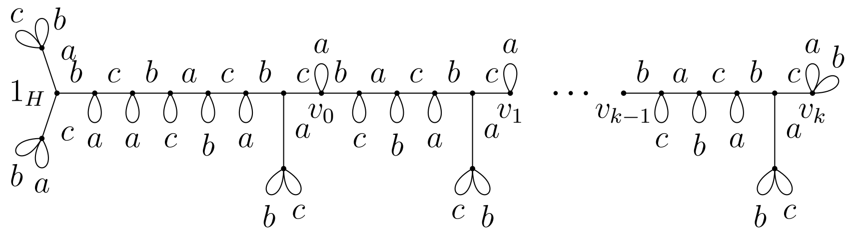

Figure 35: Graphs with $11 + 6k$ vertices, which fulfill the defining relators of the triangle group $\Delta(3l, 5m, 6n)$.

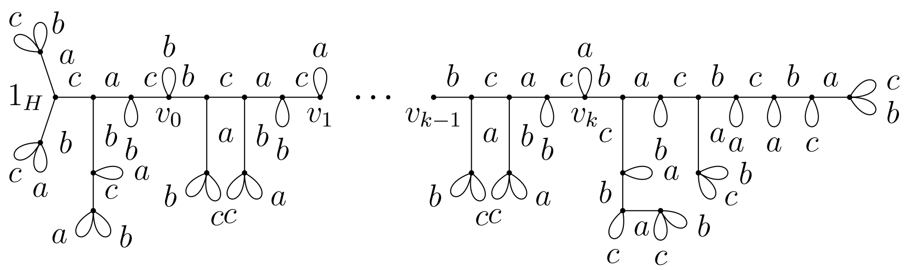

Figure 36: Graphs with $19 + 6k$ vertices, which fulfill the defining relators of the triangle group $\Delta(3l, 5m, 10n)$.

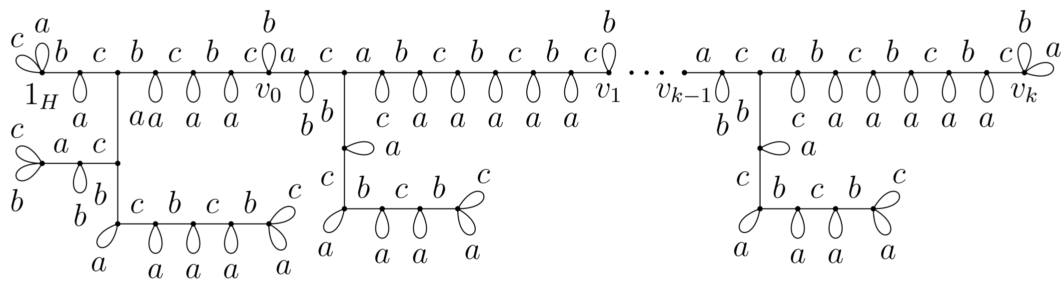

Figure 37: Graphs with $15 + 7k$ vertices, which fulfill the defining relators of the triangle group $\Delta(4l, 7m, 10n)$.

C.2 Euclidean triangle groups

In this section we consider the euclidean triangle groups $\Delta(3,3,3)$, $\Delta(2,4,4)$ and $\Delta(2,3,6)$. In Sections C.2.1–C.2.3 we prove that $\mathscr{H}_{\mathbb{P}^2}$ is infinite for all of them, which we stated in Section 6.2.3. In Section C.2.4 we consider related Coxeter groups and in Section C.2.5 we study other sequences of subgroups of $\Delta(3,3,3)$.

We prove that $\mathscr{H}_{\mathbb{P}^2}$ is infinite in the following way. We construct sequences of graphs which have a common structure each of which is built from the previous one by opening loops and adding vertices and edges. We picture the first graphs of such a sequence to clarify the structure and prove the existence. Then we determine combinatorially the number of the vertices of each graph of the sequence.

C.2.1 $\Delta(3,3,3)$

Figure 38 pictures three graphs with 9 vertices, one graph with 16 vertices, one graph with 25 vertices, and one graph with 36 vertices. The three different graphs $\Gamma_{9,ac}$, $\Gamma_{9,ab}$, and $\Gamma_{9,bc}$ are each the first graph of a sequence of graphs with n^2 vertices for $n \geq 3$. The graphs with 16 and 25 vertices are equal for each sequence and depicted in Figure 38. In fact, the graphs of the three sequences only differ if $n \in 3\mathbb{N}$. Moreover, Figure 38 pictures the graph $\Gamma_{36,ac}$.

The relators a^2, b^2, and c^2 are fulfilled, since we draw undirected edges. As in Appendix B the relators $(ab)^3$, $(bc)^3$, and $(ac)^3$ provide the following graphs: a graph with one vertex and two loops, a graph with three vertices connected with two undirected edges with different labels and a loop at both ends, and a hexagon built of undirected edges labeled alternating. Thus the graphs depicted in Figure 38 are constructed just of those graphs. Hence they are subgroup graphs of finite index subgroups of $\Delta(3,3,3)$.

Studying the graphs of Figure 38, it is easy to see that to the graph with n^2 vertices we add $2n+1$ vertices. Since $n^2+2n+1=(n+1)^2$, we proved that there exists at least one finite index subgroup of $\Delta(3,3,3)$ for any square n^2 with $n \geq 3$. Consequently, $\mathscr{H}_{\mathbb{P}^2}(\Delta(3,3,3))$ is infinite.

As a chemist the structure of the subgroups graphs in Figure 38 looks familiar. It could be a carbon compound. But as far as I know such a structure is infinite and is the graphene structure.

C.2.2 $\Delta(2,4,4)$

Figure 39 pictures the first three graphs of a sequence of subgroup graphs with $(2n+1)^2$ vertices. It is easy to see that the graphs fulfill the defining relators of $\Delta(2,4,4)$. Thus they are subgroup graphs of finite index subgroups of $\Delta(2,4,4)$. We have $9+16=3^2+3\cdot 4+4=5^2$, $25+24=5^2+5\cdot 4+4=49$. The construction is easy to understand and gives the formula n^2+4n+4 counting the vertices from Figure 39. This gives us the equation $n^2+4n+4=(n+2)^2$, which proves the claim that any graph of this sequence has square index. Since each prime bigger than 2 is odd, this proves that $\mathscr{H}_{\mathbb{P}^2}(\Delta(2,4,4))$ is infinite.

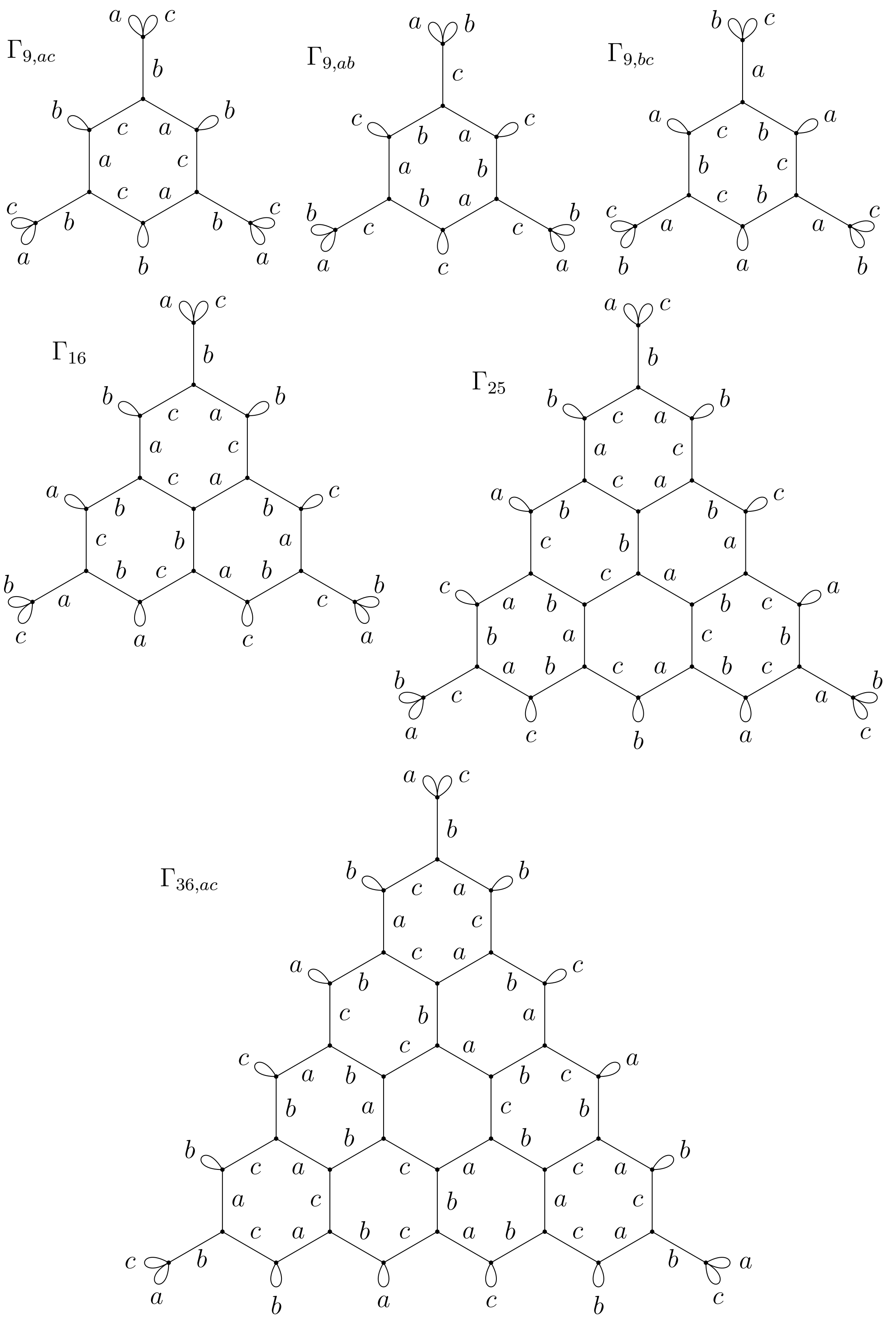

Figure 38: Subgroup graphs of subgroups of index 3^2, 4^2, 5^2 and 6^2 in $\Delta(3,3,3)$.

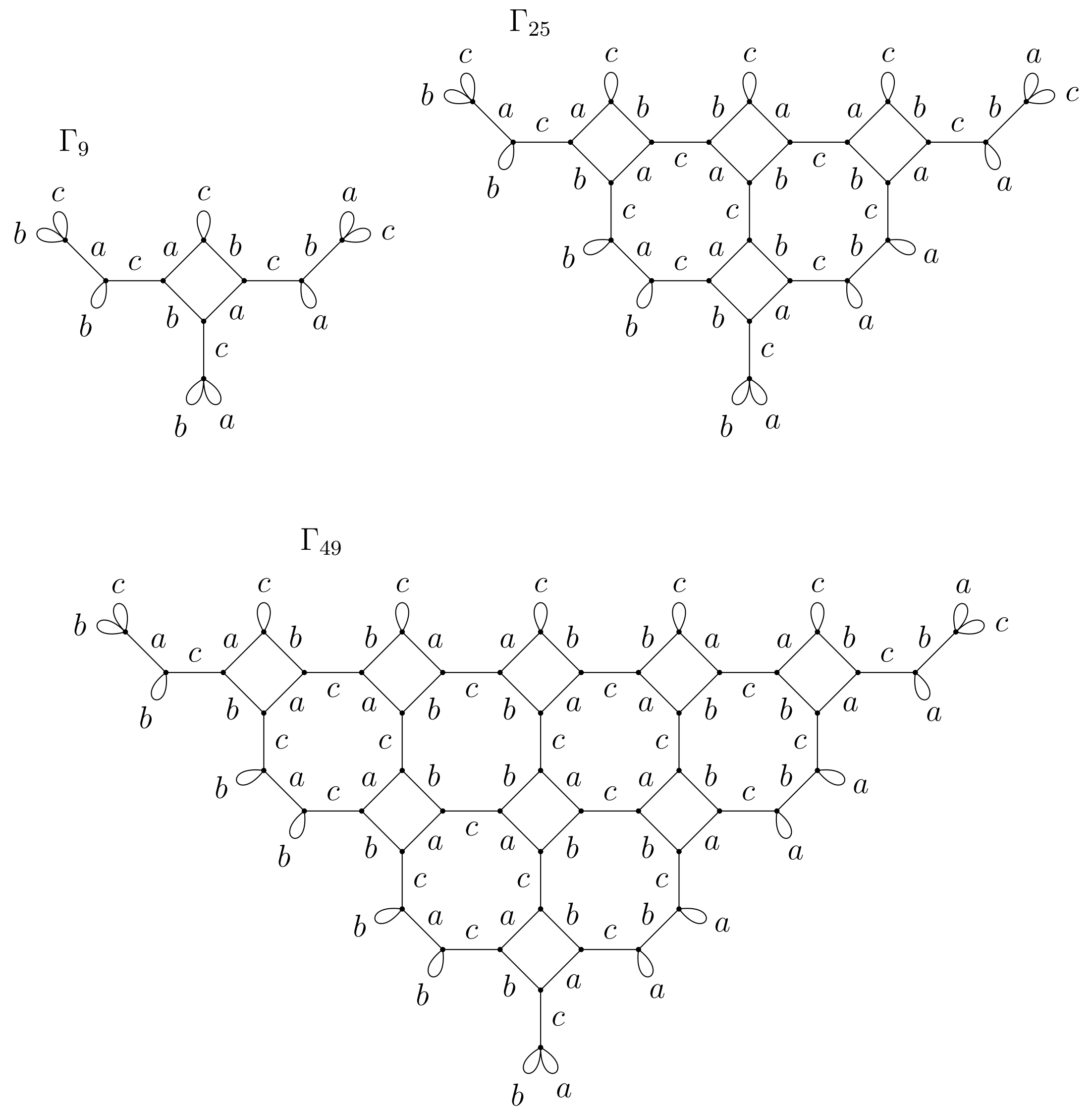

Figure 39: Subgroup graphs of subgroups of index 3^2, 5^2 and 7^2 in $\Delta(2,4,4)$.

C.2.3 $\Delta(2,3,6)$

The graphs for $\Delta(2,3,6)$ are more complicated than for the other triangle groups. Figure 40 depicts the first four graphs of a sequence of subgroup graphs with $(2n+1)^2$ vertices for $n > 0$. The graphs Γ_9 and Γ_{25} are pictured alone whereas the graph Γ_{49} is the red part of the graph Γ_{81} and the graph Γ_{81} consists of the red and the black part, except for some red loops, which become black edges.

The graphs of Figure 40 fulfill the defining relators of $\Delta(2,3,6)$. Thus they are subgroup graphs. Counting the vertices and studying the structure we get the following equations, which proves that there exist infinitely many subgroups of square prime index.

$(6k+3)^2 + 6(4n+3) - 2 = (6k+5)^2$
$(6k+5)^2 + 6(4(k+1)) = (6(k+1)+1)^2$
$(6k+1)^2 + 6(4k+1) + 2 = (6k+3)^2$

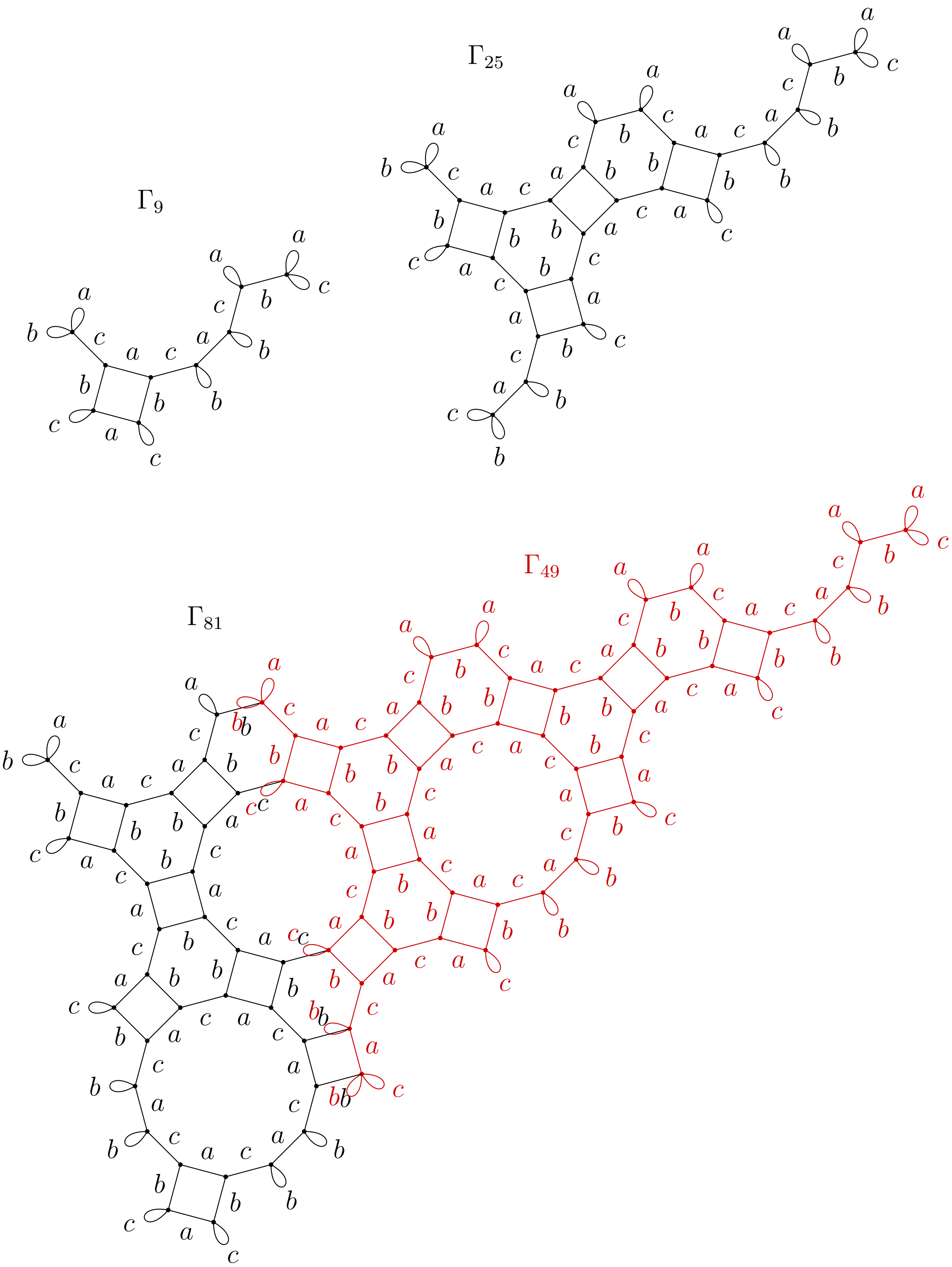

Figure 40: Subgroup graphs of subgroups of index 3^2, 5^2, 7^2 and 9^2 in $\Delta(2, 3, 6)$.

C.2.4 Related hyperbolic triangle groups and Coxeter groups

If a graph fulfills a relator r, then it also fulfills any power of this relator. Thus the Figures 38–40 prove that $\mathscr{H}_{\mathbb{P}^2}$ is infinite for the hyperbolic triangle groups

$\Delta(2l, 3m, 6n)$, $\Delta(2l, 4m, 4n)$, and $\Delta(3l, 3m, 3n)$. If we add a loop, labeled s_i, for $i = 4, ..., n$ to the graphs of Figures 38–40 we get subgroup graphs of the Coxeter groups $\langle s_1, ..., s_n \rangle$ with $(m_{1,2}, m_{2,3}, m_{1,3})$ equals $(3l, 3m, 3n)$, $(2l, 4, m, 4n)$, or $(2l, 3m, 6n)$ and $m_{1,i}, m_{2,i}, m_{3,i} \in 2\mathbb{N}_{>0} \cup \{\infty\}$ for $i = 4, ..., n$.

C.2.5 More sequences of subgroups for $\Delta(3, 3, 3)$

The triangle group $\Delta(3, 3, 3)$ has nice collections of subgroup graphs. They are depicted in Figures 41–43. The algorithm for creating the collection of graphs in Figures 41–43 is clear by observing the examples. As always the index of the graph Γ_n in the collection $\{\Gamma_n\}_{n \in N}$ is the index of the subgroup associated to (Γ_n, v) for all $v \in V(\Gamma_n)$.

Figure 41: Here $N = \{6x \mid x \in \mathbb{N}_{>0}\}$. The figure pictures the first three graphs: Γ_6 with two triangles, Γ_{12} with two triangles and one hexagon and Γ_{18} with two triangles and two hexagons. Therefore Γ_{6x} has two triangles and $(x - 1)$ hexagons.

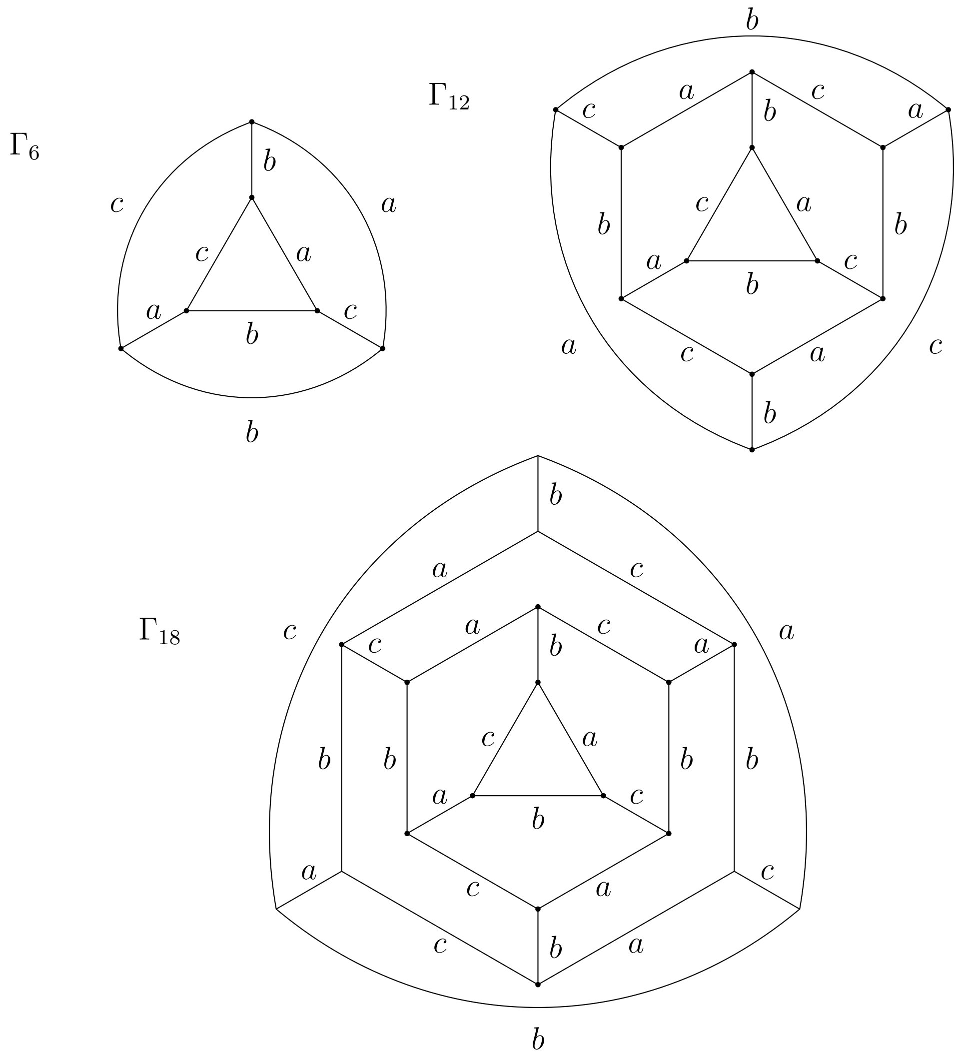

Figure 41: Subgroup graphs of subgroups of index 6, 12 and 18 in $\Delta(3, 3, 3)$.

Figure 42: Here $N = \{6x + 3 \mid x \in \mathbb{N}_{>0}\}$. The figure pictures the first three graphs: Γ_9 with one hexagon and one triangle, Γ_{15} with two hexagons and one triangle and Γ_{21} with three hexagons and one triangle. So we can see that Γ_{6x+3} has x hexagons and one triangle.

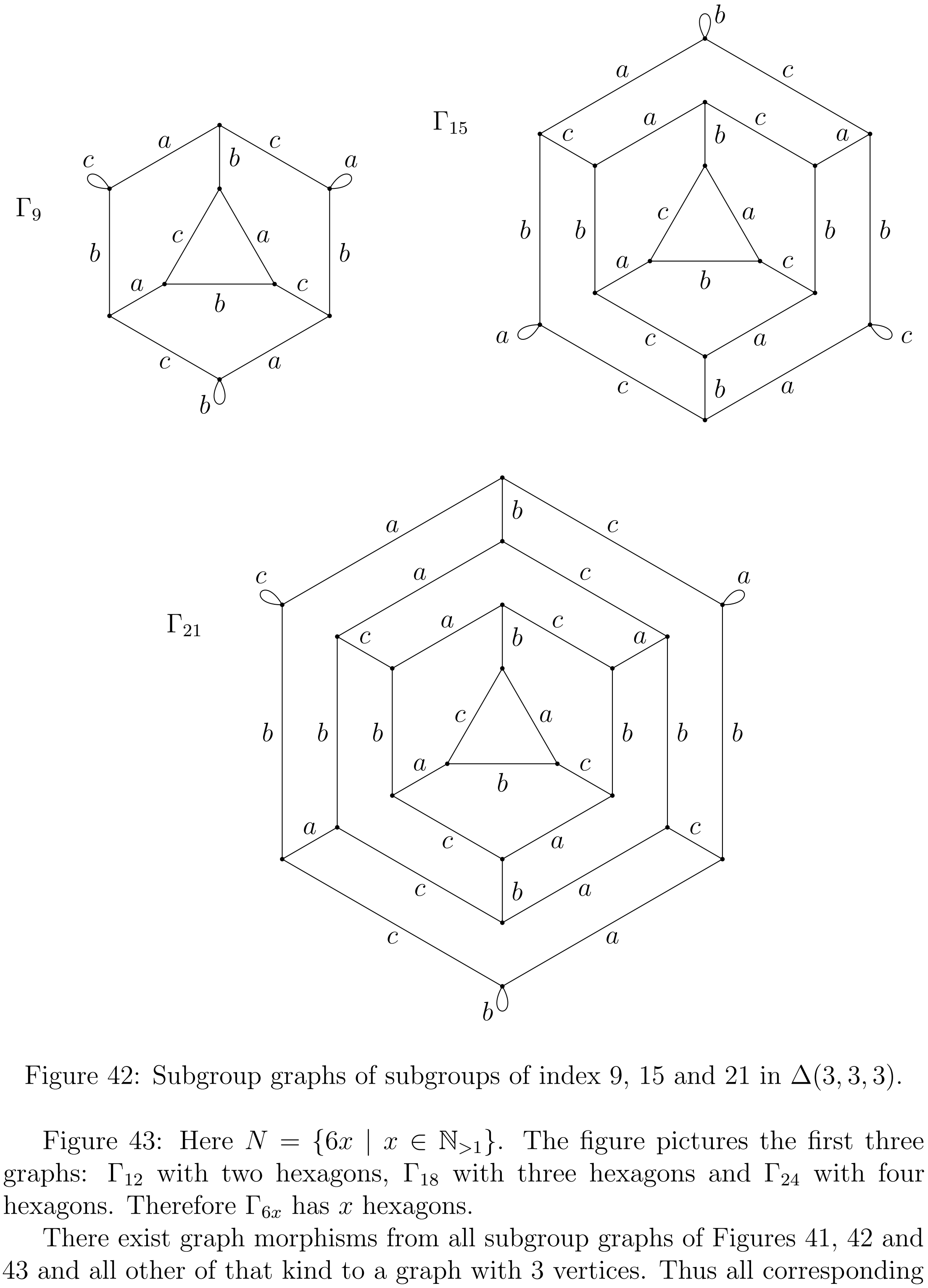

Figure 42: Subgroup graphs of subgroups of index 9, 15 and 21 in $\Delta(3, 3, 3)$.

Figure 43: Here $N = \{6x \mid x \in \mathbb{N}_{>1}\}$. The figure pictures the first three graphs: Γ_{12} with two hexagons, Γ_{18} with three hexagons and Γ_{24} with four hexagons. Therefore Γ_{6x} has x hexagons.

There exist graph morphisms from all subgroup graphs of Figures 41, 42 and 43 and all other of that kind to a graph with 3 vertices. Thus all corresponding subgroups are not maximal subgroups.

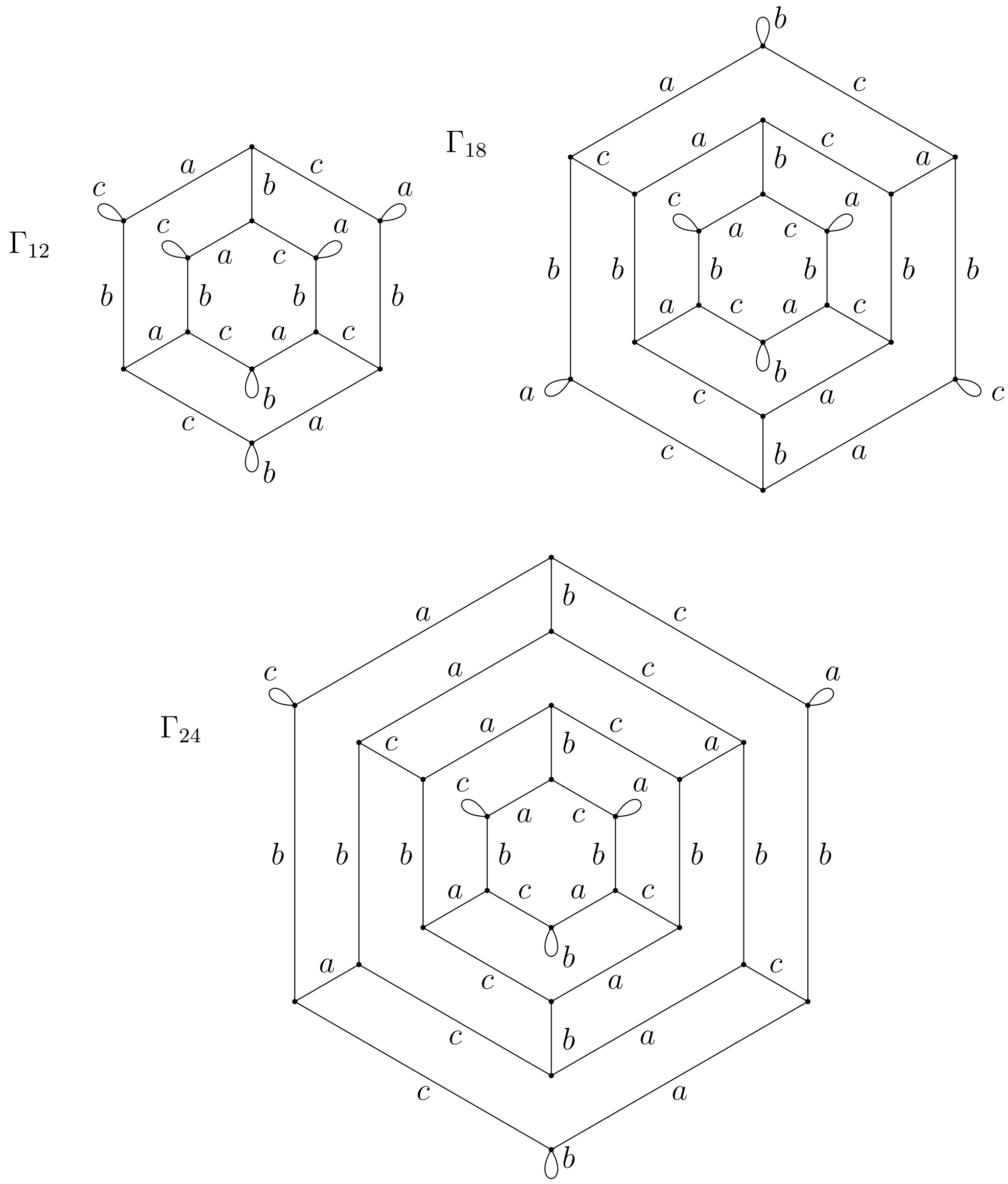

Figure 43: Subgroup graphs of subgroups of index 12, 18 and 24 in $\Delta(3,3,3)$.

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